Transforming Parabolas

1 All Parabolas Come From One

The graphs of functions of the form $f(x) = ax^2 + bx + c$ (quadratic functions) are called parabolas (parabolas can come up for other relations, $y^2 = px$ being another classic case). If you have played around a little with a graphing tool, you have a good idea of their shape. Here, we want to show how they can always be obtained from the "basic" graph of $y = x^2$, by a short sequence of simple transformations.

1.1 Which Way Is Open?

We want to transform from curve B, $y = x^2$, to curve A, $y = ax^2 + bx + c$. First, we look at the coefficient a of parabola A. If it is positive, we move on, if it is negative, we start by flipping the graph of B, into $-x^2$.

1.2 How Open Is Open?

We know that the coefficient a determines, with its absolute value, how open the parabola is. Hence, we multiply our new graph B by $|a|$, so that our equation is now $y = ax^2$ (we already multiplied by the sign of a before - we could have condensed it all in one step, of course).

This is the trickiest step geometrically. The idea is to shrink or stretch the graph, until they look identical, except they are located in different parts of the plane.

1.3 Locate The Vertex

Parabola A has vertex at $x = -\frac{b}{2a}$. Let us *shift horizontally* our latest B by this amount. This means that, in the new equation, the height y that corresponded to position x, now corresponds to the position where x was before the shift. In other words, we have to substitute, in the equation for B, $x - \left(-\frac{b}{2a}\right) = x + \frac{b}{2a}$, for x. B now has equation $y = a\left(x + \frac{b}{2a}\right)^2$.

Geometrically, we now have two identical parabolas, aligned, with the same axis, but shifted vertically from one another.

1.4 Bring The Vertexes Together.

We now have to bring B to A, by shifting it vertically by the appropriate amount. We can do that in two ways:

1. Parabola A intersects the y axis in $y = c$. We find where B now intersects the y axis (that's when $x = 0$: the intersection is at $y = a\frac{b^2}{4a^2} = \frac{b^2}{4a}$. Now, shift B vertically by the difference of the y intercepts, $c - \frac{b^2}{4a}$.

2. Parabola A has its vertex at

$$y = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = \frac{b^2}{4a} - \frac{b^2}{2a} + c = -\frac{b^2}{4a} + c$$

   and parabola B has it at $y = 0$. Just shift B by this amount, the same as in point 1.

1.5 Summing Up

We transform from $y = x^2$ to $y = ax^2 + bx + c$ by

1. Stretching: multiply the function by a: $\to y = ax^2$
2. Shifting horizontally by $-\frac{b}{2a}$: $\to y = a\left(x + \frac{b}{2a}\right)^2$
3. Shifting vertically by $c - \frac{b^2}{4a}$: $\to y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$

Of course, we have, indeed

$$a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = a\left(x^2 + \frac{b^2}{4a^2} + 2x\frac{b}{2a}\right) + c - \frac{b^2}{4a} =$$

$$ax^2 + \frac{b^2}{4a} + bx + c - \frac{b^2}{4a} = ax^2 + bx + c$$

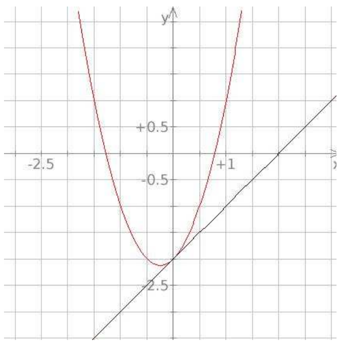
2 What Do Those Coefficients Mean?

You can say a lot about a parabola, straight from its equation, without calculating anything. Specifically, each of the coefficients, a, b, c has a specific direct geometric meaning.

c This we already know: it tells us where the $y-$intercept is.

b This is more subtle. We have to observe that, when a number is less than 1, its square is much less than the number itself. This means that, as we look at our parabola for values of x extremely close to 0, the term ax^2 will be extremely smaller than the remaining two, $bx + c$. Our parabola's equation will then look very much like the graph of $y = bx + c$. But this is the graph of a line, intersecting the y axis at $y = c$ (just like our parabola),

and with slope b. Thus the line that "almost" coincides with the parabola near $x = 0$ has slope b.
This line is called the *tangent line* to the parabola at $x = 0$, and you certainly know what it is, in a picture:



Note how we have the following cases:

- $b > 0$, the parabola is increasing as it crosses the y axis
- $b = 0$, the parabola is neither increasing, nor decreasing as it crosses the y axis - it's got to be the vertex!
- $b < 0$, the parabola decreases as it crosses the y axis.

You can combine this with the quick observation about which way the parabola is open, to find that we have these four cases:

1. $a > 0, b > 0$: Open up, crosses the y axis increasing, hence, the vertex *must be* over negative x's
2. $a > 0, b < 0$: Open up, crosses the y axis decreasing, hence, the vertex *must be* over positive x's.
3. $a < 0, b > 0$: Open down, crosses increasing - vertex has to be on the positive x axis
4. $a < 0, b < 0$: Open down, crosses decreasing - vertex has to be on the negative axis

If you are willing to remember the formula $-\frac{b}{2a}$ for the x coordinate of the vertex, the same conclusion follows simply by observing its sign, depending on the relative signs of a and b.

a We already know what the sign of a means. We also have a qualitative idea of the meaning of its value - the higher it is, the sharper, narrower, the graph looks. But is there a way to give a precise meaning to this value - like c giving an intercept, and b giving the slope of a tangent line? There is, and, unfortunately it's way beyond the scope of our course to prove the following fact - but it may be interesting to know, nonetheless:

Consider the parabola near its vertex. It has a rounded shape, similar to - but different from - a circle. We could ask for the circle that best approximates the parabola at its vertex - after all, circles are geometrically easier to visualize. If we look for a circle, with its center on the axis of the parabola, going through the vertex point, and which most closely tracks the graph of the parabola very near its vertex, we find that its radius depends on a only. In fact, this radius (called *the curvature radius of the parabola*, at its vertex, is

$$R = \frac{1}{2a} \tag{1}$$

2.1 A proof of (1)

If you really want to know, here is how the argument works. Let's stick with the basic parabola $y = ax^2$, with $a > 0$. It is easy to see how the argument will work in any other case.

A circle with center on the symmetry axis, $x = 0$, of the parabola, will have equation (a circle is the set of points at fixed distance from the center – now, apply the Pythagorean Theorem, calling the center $(0, c)$)

$$x^2 + (y - c)^2 = r^2$$

Since we want it to go through the point $(0, 0)$, we have $c^2 = r^2$, and the equation is $x^2 + (y - r)^2 = r^2$, or

$$x^2 + y^2 - 2yr = 0$$

Now we want to find the intersections with the parabola $y = ax^2$. Substituting:

$$x^2 + a^2x^4 - 2arx^2 = 0$$

One solution is the obvious (double) one, $x^2 = 0$. The others are given by factoring

$$a^2x^2 + 1 - 2ar = 0$$

$$x^2 = \frac{2ar - 1}{a^2}$$

This will both coincide with the trivial one, making it a multiple (four-fold) intersection ("as close as possible") if

$$2ar = 1$$

that is

$$r = \frac{1}{2a}$$