Graphing Polynomial Functions

Calculus without calculus

1 Introduction

You may want to check out the PDF file "Average Rate of Change for Quadratic Functions", where we discussed how the average rate of change of a Polynomial function of degree n turns out to be a polynomial itself, of degree $n-1$. What's more, even if demonstrated only on a quadratic function, we saw that, when we compute the average rate of change over short intervals, the result differs very little from the slope of the tangent line to the curve.

All this sets us up to tackle one big part of the problem of describing polynomials in more detail: it allows us to find where these functions are increasing, where they are decreasing, and where they have turning points. Since one of the tools is the solving of algebraic equations, the results are not as all-solving as we would like: to study, say, a 6th degree polynomial, we would need to solve 6th and 5th degree equations, which is a formidable task, if we need thee solutions to be explicit and exact. Still, it makes polynomials the easiest "complicated" functions to study, and it also provides a blueprint for the study of even more complicated functions, as developed in Calculus.

2 Finding Zeros

If you have a polynomial you want to study, you could start with the location of its zeros - preferably called the "roots" of the polynomial. Unfortunately, this is usually impossible to do in exact form: we only have general solution formulas for equations of degree up to four (and those for degree 3 and 4 are very cumbersome). Given the ease with which we can compute with polynomials, there are very good approximation methods to find those roots (they are usually coded in mathematical software for quick use).

One problem where these methods are not necessarily adequate is in deciding whether two roots are extremely close or they coincide. Numerical methods have a hard time distinguishing 0 from very small numbers. Of course, in most practical applications, this is not a big problem, but it can be in special circumstances.

It is often not very hard to come up with a ballpark figure for the roots, and the book lists several tools for that. One crucial property polynomials have is to have a "continuous" graph. The intuitive meaning of this is that the graph does not exhibit jumps or gaps. What this, again intuitively, entails is that if a polynomial is positive at, say, $x = x_1$, and negative at $x = x_2 > x_1$, it will have to have at least a root between these two points[1].

3 Points of Increase and Points of Decrease

If you look at the graph of a polynomial, as presented in a book or by a software package, you will notice that it has a simple pattern: "coming from" very negative values of the independent variable, it will be either increasing or decreasing; the rate at which it is doing so will slow down as you move on; this may go on forever, with the rate possibly going up and down; in most cases, though, at some point the graph will be practically horizontal, after which it may start to increase (or decrease) again, at an accelerating pace, or, more commonly, it may switch direction; this sequence may repeat a number of times, until the graph begins to increase, or decrease, at an accelerating pace for the rest of its domain.

Much of this behavior can be captured directly from the formula expressing the polynomial (after all, that's what the software is doing anyway), and we can start looking for the tools to do so.

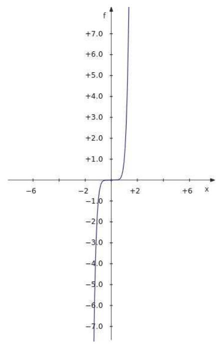
To fix ideas, let us keep as a guinea pig a polynomial with no special apparent simplifying features, like

$$p(x) = x^7 - 3x^6 + 5x^4 - 2x^3 - x^2 + 5x + 4$$

3.1 Tail ("End") Behavior

As discussed in the book, when $|x|$ is very large, the behavior of a polynomial is almost indistinguishable from that of the power function equal to its highest degree term. Since this behavior is easy to describe, we have an immediate way of seeing what is going on at the two ends of the graph. In the case of our guinea pig, this term is x^7, a high odd power, with a positive coefficient. The graph is somewhat similar to the cubic function x^3, with much steeper growth:

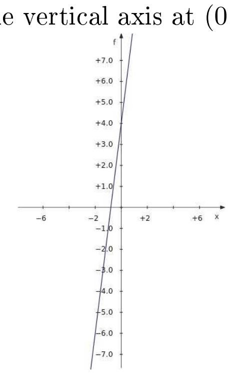
[1] Since in Math we want to extend our scope to much more complicated functions, both the notion of being "continuous", and the fact that such a function needs to take the value zero between two points where it has opposing signs, need to be cast in a much more abstract and precise form. You will deal with this more general approach in Calculus - but the point is that the rigorous treatment is there so that at least some of the easy things we can do with polynomials can be extended to more complicated cases.



3.2 Crossing the $y-$axis

This is also an easy task. We look at the constant term, and that's the $y-$intercept. Also, the linear term tells us about the slope with which the axis is crossed.

In our example, the two lowest terms are $5x + 4$, hence the graph will cross the vertical axis at $(0, 4)$, with a tangent whose slope is 5. That's pretty steep:



3.3 Increasing or Decreasing? Turning Points?

This is much harder. By following the indications of the file "Average Rate of Change for Quadratic Functions", we see that the slope of the tangent at a point x will be given by the function

$$s(x) = 7x^6 - 18x^5 + 20x^3 - 6x^2 - 2x + 5$$

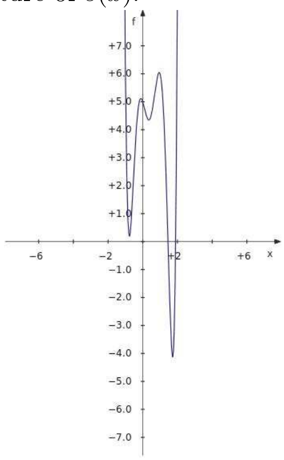
which is just as hard to study as our original. Of course, we know that for large $|x|$, $s(x)$ is positive, and so the graph of $p(x)$ will be increasing - but we already knew that!

One thing we can do is put in some values for x, and check whether s is positive or negative. This will point us towards areas where p is increasing or decreasing. Now, since we know that $s(x)$ has at most six roots, there are at most six points at which it will change sign - which is where p will have a turning point.

Here is a little table, courtesy of a helpful software package:

x	$s(x)$
-1	6
1	6
1.5	-0.95312
1.8	-3.4367
2	9

We see that there must be a zero between 1 and 1.5, and another between 1.8 and 2. In fact, here is what a graphing utility will give us if we ask for a picture of $s(x)$:



This suggests that $p(x)$ will be almost always increasing, except for a short interval between 1 and 2, when it will be decreasing. It will have a "peak" (a maximum) at the beginning, and a "valley" (a minimum) at the end of this interval.

Here is the graphing utility version of $p(x)$: