Long Division

"Division" between positive integers n and $d \neq 0$ means finding two positive integers q and r, with $r < d$, such that

$$n = q \cdot d + r$$

The condition $r < d$ ensures that there is only one solution to this problem.

"Division" between polynomials $n(x)$ and $d(x)$ means finding two polynomials,$q(x)$ and $r(x)$, with the degree of $r(x)$ less than the degree of $d(x)$, such that

$$n(x) = q(x)d(x) + r(x)$$

Again, the condition on the degree of r ensures that there is only one solution to this problem. Clearly, we'll have

$$\deg q + \deg d = \deg n$$

Also, only the highest degree in q and in d will contribute to the highest degree n n. It is natural, then, to start by concentrating on the leading terms.

Thus, if $n(x) = ax^k + \ldots$, and $d(x) = bx^h + \ldots$, with $h \leq k$, we start by computing

$$\frac{ax^k}{bx^h} = \frac{a}{b}x^{k-h}$$

This will be the leading term of the quotient. Multiplying this term by $d(x)$ and subtracting from $n(x)$ will allow us to find that

$$\frac{n(x)}{d(x)} = \frac{a}{b}x^{k-h} + \frac{n'(x)}{d(x)}$$

where the degree of n' is at most $k-1$. We can repeat the process with this new fraction. The procedure stops when we end up with a numerator with degree less than h.

Here's a concrete example to illustrate how it works.

Take

$$\frac{3x^4 - 2x^3 + x^2 - x - 2}{x^2 + 3x + 1}$$

We start working toward the result by looking at the highest powers:

$$\frac{3x^4}{x^2} = 3x^2$$

This is certainly not the desired $q(x)$, but it will be its leading term, and we can check how far we are form our goal: we compare $3x^2 \cdot d(x)$ with $n(x)$, and the difference will be our "error term".

$$3x^2 \cdot \left(x^2 + 3x + 1\right) = 3x^4 + 9x^3 + 3x^2$$

Subtract from the numerator:

$$3x^4 - 2x^3 + x^2 - x - 2 - \left(3x^4 + 9x^3 + 3x^2\right) = 0 \cdot x^4 - 11x^3 - 2x^2 - x - 2$$

Let's write explicitly what this result means:

$$\frac{3x^4 - 2x^3 + x^2 - x - 2}{x^2 + 3x + 1} - \frac{3x^2 \cdot \left(x^2 + 3x + 1\right)}{x^2 + 3x + 1} = \frac{-11x^3 - 2x^2 - x - 2}{x^2 + 3x + 1}$$

or

$$\frac{3x^4 - 2x^3 + x^2 - x - 2}{x^2 + 3x + 1} = 3x^2 + \frac{-11x^3 - 2x^2 - x - 2}{x^2 + 3x + 1}$$

The difference is thus a new fraction, with a numerator that's got a lower degree. We can repeat now the process on this new fraction. This will yield a second term. Repeating all steps, we keep going until we are left with a polynomial whose degree is less than that of the denominator. This is the "remainder" $r(x)$.

The usual way of writing long division is just a mnemonic to automate this process. The drawback is that we easily lose sight of what we are really doing.

There Is Another Way

The ability to write a rational function as "polynomial + rational function with a numerator of lesser degree than the denominator" turns out to be useful in Calculus, as this form is easier to handle when computing integrals (you'll see what they are, and why you should care, in that class)

However, I suspect that most people will not really use long division for this purpose. There is an equivalent, and less artificial way of obtaining the same outcome. Suppose you want to divide a polynomial, I'll take, as an example, $2x^3 + 3x^2 - 2x + 1$, by another polynomial of lower degree, and I'll take $x - 5$ as an example. How can you do it? You can use long division or synthetic division (because I chose a divisor like $x - 5$ - it would not be available for something like $x^2 - 2x - 1$). However, there's a simpler way.

To explain how it works, let me remind you what "division of A by B results in C with remainder D" means (this is true for whole numbers, and it is true for polynomials): it means that

$$\frac{A}{B} = C + \frac{D}{B}$$

but, even better, it means that

$$A = B \cdot C + D \tag{1}$$

Now, it will turn out that

$$\frac{2x^3 - 3x^2 - 2x - 1}{x - 5} = 2x^2 + 7x + 33 + \frac{164}{x - 5}$$

that is,

$$2x^3 - 3x^2 - 2x - 1 = (x - 5)\left(2x^2 + 7x + 33\right) + 164$$

We can guess immediately that the quotient will be of degree 2, since, when multiplied by $x - 5$ it must result in something of degree 3. We also guess from the start that the remainder is a number, because it has to be of a lower degree than the denominator. So, let's just write that there must be some polynomial of degree two. and a number that work as in (1) – we just don't know the specific values of the coefficients:

$$2x^3 - 3x^2 - 2x - 1 = (x - 5)\left(ax^2 + bx + c\right) + d$$

Now, let's expand the right hand side:

$$2x^3 - 3x^2 - 2x - 1 = ax^3 + (b - 5a)\,x^2 + (c - 5b)\,x - 5c + d \tag{2}$$

If the two sides have to be equal, then the coefficients of each power must be the same. That is

$$a = 2$$

$$b - 5a = b - 10 = -3$$

that is $b = 7$

$$c - 5b = c - 35 = -2$$

that is $c = 33$

$$d - 5c = d - 165 = -1$$

that is $d = 164$.

We have shown that, indeed,

$$2x^3 - 3x^2 - 2x - 1 = (x - 5)\left(2x^2 + 7x + 33\right) + 164$$

If the polynomials are big this can take some time, but, except for some care in doing the multiplication (that is obtaining (2)), the calculations are trivial:

as you see, the first coefficient is given right away, and the others follow in a cascade.

Here is another example:

$$\frac{2x^3 + 3x^2 - x - 5}{3x^2 + 4x + 1}$$

Since the denominator is of degree one less than the numerator, the quotient will have to be of degree 1. The remainder, if any, will have to be of degree less than the denominator, hence it will also be (at most) of degree one. Now, we don't know what these two linear functions will be, but they will be of the form $ax + b$, and $cx + d$. By definition, they must be such that

$$2x^3 + 3x^2 - x - 5 = (ax + b)\left(3x^2 + 4x + 1\right) + cx + d$$

Now, if we expand the right hand side, and match the coefficients of the corresponding powers on the left hand side, we find

$$2x^3 + 3x^2 - x - 5 = 3ax^3 + 4ax^2 + ax + 3bx^2 + 4bx + b + cx + d$$

$$2x^3 + 3x^2 - x - 5 = 3ax^3 + (4a + 3b)\, x^2 + (a + 4b + c)\, x + b + d$$

$$2 = 3a \Rightarrow a = \frac{2}{3}$$

$$3 = 4a + 3b = \frac{8}{3} + 3b \Rightarrow b = 1 - \frac{8}{9} = \frac{1}{9}$$

$$-1 = a + 4b + c = \frac{2}{3} + \frac{4}{9} + c \Rightarrow c = -1 - \frac{10}{9} = -\frac{19}{9}$$

$$-5 = b + d = \frac{1}{9} + d \Rightarrow d = -5 - \frac{1}{9} = -\frac{46}{9}$$

and the quotient is $\frac{2}{3}x + \frac{1}{9}$, while the remainder is $-\frac{1}{9}\,(19x + 46)$