More About Logarithms

Everything we can do with logarithms is due to their properties as exponents. recall:

$$a^x = b \quad \Leftrightarrow \quad x = \log_a(b)$$

(works for any $a, b > 0$, $a \neq 1$, and any real x).

Important consequences (expressing that exponentials and logarithms are inverse functions of each other) are

- $a^{\log_a(b)} = b$
- $\log_a(a^x) = x$

Given the properties of powers,

$$a^x a^y = a^{x+y}, \quad (a^x)^y = (a^y)^x = a^{xy}$$

and so on, we have corresponding properties for logarithms:

$$\log_a(b \cdot c) = \log_a(b) + \log_a(c), \quad x \cdot \log_a(b) = \log_a(b^x) \tag{1}$$

and so on (as listed in the book, in additional material, and everywhere else).

There are some consequences that are particularly useful. For example,

- $\log_a(1) = 0$
- $\log_a(a) = 1$
- $\log_a\left(\frac{1}{b}\right) = -\log_a(b)$
- If $b = a^x$, that is $x = \log_a(b)$, then $b^y = \left(a^{\log_a(b)}\right)^y = a^{y \cdot \log_a(b)}$
- From that same equation, it follows that if we denote $c = b^y$, that is $y = \log_b(c)$

$$\log_a(c) = \log_b(c) \cdot \log_a(b)$$

$$\log_b(c) = \frac{\log_a(c)}{\log_a(b)} \tag{2}$$

- As a consequence (set $c = a$)

$$\log_b(a) = \frac{1}{\log_a(b)} \tag{3}$$

The last two points prove the "change of base" formula: we can decide to use one base only, and every exponential or logarithm that comes up in a different base, can always be rewritten in terms of our preferred choice.

It turns out that, at least as far as the study of exponential and logarithmic *functions*, the most convenient base is the number e, as strange as that my seem. The full advantage is hard to explain without a little calculus, so we might just take this as a fact.

On the other hand, formulas like (1) used to be the means by which complicated calculations could be made by hand, before the arrival of calculators and personal computers: given a table that lists the logarithm in some base of "any" number, and, read in reverse, finds "any" number, once you know its logarithms, it became possible to calculate difficult things like, say, $\sqrt[5]{126.7}$: from (1)

$$\log_a\left((126.7)^{\frac{1}{5}}\right) = \frac{1}{5} \cdot \log_a(126.7)$$

and a fifth root is reduced to doing a division (a much simpler calculation). When dealing, like here, with fixed numbers, rather than functions, since we use decimal notation for numbers (and, in the sciences, this is reinforced by scientific notation), using base 10, instead of e can be convenient. That's how *decimal* logarithms are present in Chemistry (pH), geology (the Richter scale for earthquakes), acoustics (decibels are decimal logarithms), and so on.

In our course, decimal logarithms are simply denoted by log (no subscript, 10 is implied), and logarithms in base e ("natural logarithms") are denoted by ln.

Logarithmic Expressions

Properties of logarithms make it possible to simplify expression. From the connection between exponentials and logarithms in the same base (see above), we can see immediately that, for example

$$\log_2\left(2^{4x+1}\right) = 4x + 1$$

But we can also see that

$$\log_2\left(8^{1-3x}\right) = \log_2\left[\left(2^3\right)^{1-3x}\right] = \log_2\left(2^{3(1-3x)}\right) = 3 - 9x$$

Actually, it works, at least in part, even when the relation between the two bases is not as simple:

$$\log_2\left(3^{4-x}\right) = \log_2\left[\left(2^{\log_2(3)}\right)^{4-x}\right] = \log_2\left(2^{\log_2(3)\cdot(4-x)}\right) = \log_2(3) \cdot 4 - x$$

Similarly for the "reverse" case, as in $2^{\log_2(x+1)} = x + 1$, or $8^{\log_2(x+1)} = \left(2^3\right)^{\log_2(x+1)} = (x+1)^3$

Logarithmic Equations

The idea is that if $\log_a [f(x)] = \log_a [g(x)]$, then it must be that $f(x) = g(x)$. If two logarithms *in the same base* are equal, then their arguments must be equal. Hence, the trick is to reduce a logarithmic equation to this form.

For example
$$\log_2(x^2+1) - 2\log_2(x+1) = 1 \tag{4}$$
since $2\log_2(x+1) = \log_2\left[(x+1)^2\right]$, and $1 = \log_2(2)$ is equivalent to
$$\log_2\left[\frac{x^2+1}{(x+1)^2}\right] = \log_2(2)$$
that is
$$\frac{x^2+1}{(x+1)^2} = 2$$
which leads to a quadratic equation,
$$x^2+1 = 2x^2+4x+2$$
$$x^2+4x+1 = 0 \tag{5}$$

If there is more than one base, we change things so that only one base appears, relying on (2). For example, $4 = 2^2$, or $2 = 4^{\frac{1}{2}}$. Hence, $\log_2(4) = 2$, and $\log_4(2) = \frac{1}{2}$ (compare with (3))

Thus, faced with, say
$$\log_4\left(3x^2+2\right) - \log_2(x+1) = 2 \tag{6}$$
we can set everything to base 2, using
$$\log_4\left(3x^2+2\right) = \frac{\log_2\left(3x^2+2\right)}{2} = \frac{1}{2}\log_2\left(3x^2+2\right) = \log_2\left(\sqrt{3x^2+2}\right)$$
$$2 = \log_2(4)$$
leading to
$$\frac{\sqrt{3x^2+4}}{(x+1)} = 4$$
$$\sqrt{3x^2+4} = 4x+1$$
and a quadratic equation after squaring,. But we can also use
$$\log_2(x+1) = \frac{\log_4(x+1)}{\frac{1}{2}} = 2\log_4(x+1) = \log_4\left[(x+1)^2\right]$$
$$2 = \log_4(16)$$
leading to the same quadratic equation:
$$\frac{3x^2+4}{(x+1)^2} = 16 \tag{7}$$

Careful!

The equation you end up with, starting from a logarithmic one, might not be fully equivalent to the original one. what could happen is that all solutions to the logarithmic equation will be solutions to the one you ended up with, but the reverse might not be true, because solutions to the final equation may fall outside the domain of the original one.

That means that, once you solve your equation, you must make sure your solutions are in the domain of the original equation. If they are not, they are *spurious solutions* and must be discarded.

Looking for example, at equation (4), its domain is determined by $x^2+1>0$ (that's given by all real numbers), and $x+1>0$, that is any proper solution must satisfy $x>-1$. The solutions to (5) are

$$x = 2 \pm \sqrt{3}$$

which are both positive, hence they are both legitimate. You can check that the same argument allows us to accept both solutions to (7) as proper solutions to (6).