More On Exponentials And Logarithms

1 Transformations Of Exponentials

We studied some transformations that change a parabola into another. If you recall, this had us change from a function $f(x)$, to a transformed function $kf(bx+c)+d$.

When it comes to exponentials, it turns out that some of these transformations are actually one and the same. We ignore vertical shifts here, as they do what they are expected to do, and look at the remaining ones.

Aa^{bx+c} means, taking $e(x)=a^x$, and going to $Ae(bx+c)$. I.e., we are

1. Changing the horizontal scale ($x \to bx$)
2. Shifting by $-c$ ($bx \to bx+c$)
3. Multiplying by a constant, that is changing the vertical scale.

However, the operations can also be read differently.

1. Changing base: k^x, if $k=a^b$ (and that's OK for any $k>0$),

$$k^x = \left(a^b\right)^x = a^{bx}$$

(same as changing horizontal scale)

2. Change the vertical scale: $e(x) \to Ae(x)$, if $A=a^c$ (that's always true, for some c, as long as $A>0$),

$$Aa^x = a^c a^x = a^{x+c}$$

(same as shifting horizontally!)

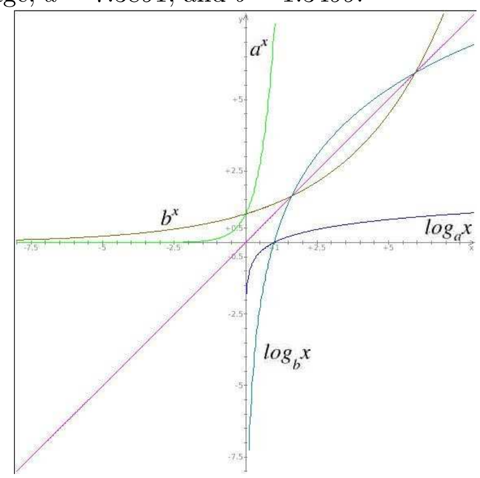
2 Some Simple Comparisons

1. $x>0, \quad a>b \Rightarrow a^x > b^x$ (let $a=cb$, with $c>1$, so that $a^x=(bc)^x = b^x c^x$, and $c^x>1$, if $x>0$)
2. $x<0, \quad a>b \Rightarrow a^x < b^x$ (it goes like before, but now, if $x<0$, and $c>1$, $c^x<1$).

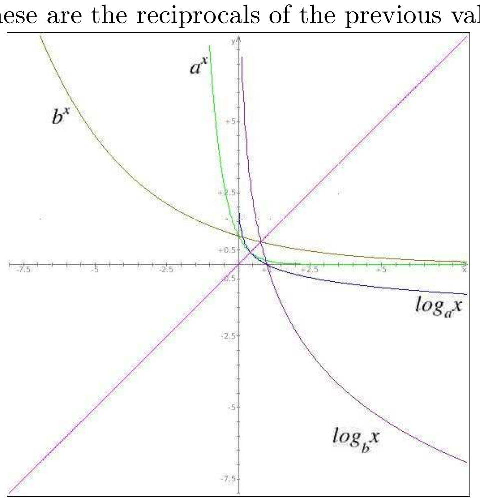
3 Graphs and Reflections

Note that, for any a, $\log_a 1 = 0$, just because, for any a, $a^0 = 1$ (we recall that we are only considering positive bases, $a > 0$ - also, given that $1^x = 1$ for any x, we consider $a \neq 1$, as that is a trivial case).

Looking now at the graph of exponentials and logarithms, we have a few observations. Since for $a > 1$, a^xgrows very fast, $\log_a x$ grows very slow. Of course, a lot also depends on the value of a. The following picture shows graphs for a^x, b^x and $\log_a x, \log_b x$, with $a, b > 1$. Precisely, in the graph on the following page, $a = 7.3891$, and $b = 1.3499$.



The behavior is similar, essentially involving a reflection around an axis, when $a, b < 1$: the graph on the following page is for $a = 0.1534$, $b = 0.74082$ (these are the reciprocals of the previous values)



In fact, take $c < 1$. Since $c < 1$, $\frac{1}{c} = k > 1$. Now, the graph of c^x is the graph of $\left(k^{-1}\right)^x = k^{-x}$ - that is the graph of k^x, reflected around the $y-$axis (x

goes into $-x$). Similarly, we can compare $\log_c x$ with $\log_{k^{-1}} x = \log_k x$:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

by the "change of base" formula. But $\log_a b = \log_{b^{-1}} b = -1$ (since $\left(b^{-1}\right)^{-1} = b$), so that

$$\log_b x = -\log_a x$$

or, the graph of $\log_b x$ is the graph of the opposite of $\log_a x$ - that is a reflection around the $x-$ axis (y goes to $-y$).

As noted in the book, since exponentials and logarithms are functions inverse of each other, their graphs are related: one is the reflection of the other across the diagonal line $y = x$, which is also drawn in the pictures.

3.1 * When do the graphs of u^x and $\log_u(x)$ intersect?

Disclaimer: This is a totally unnecessary section. It is here because *I* got curious about this question, found the answer, and thought I'd write it down, just in case anybody was wondering too... It is more than optional: it's only for those with too much time on their hands, and an odd curiosity they'd like to satisfy.

If you look at the first graph in sec. 3, you'll notice that the graphs of a^x, and $\log_a x$ do not intersect, while those of b^x, and $\log_b x$ do (in two points). It is fairly clear that if you look at the graph of an exponential like u^x, for $u > 1$, it will get lower and lower, as you decrease u. Since the graph of $\log_u x$ is the mirror image around the line $y = x$, it will do exactly the opposite. So, there will be a largest value u, such that they intersect. For values for which they do intersect, those intersections, by symmetry, will be on the line $y = x$, and there will be generally two of them, except for that largest value of u for which only one intersection exists: at that value of u, the two graphs will only touch in one point, which will lie on the line $y = x$. All of this could be proved rigorously, but it is pretty clear to see it, if you just picture how those graphs change, as you change u (you could also play around with a graphing calculator or graphing program).

Can we compute exactly what this special value of u is? At first glance it's not that easy, but on second thought we can. While a full-fledged rigorous calculation would need some calculus, we can make do with just one intuitive notion that was presented in the companion file "Bases for Logarithms". There, we argued that an exponential function e^{mx} has a rate of change at a point (h, k) proportional to its value at that point, $me^{mh} = mk$. For $u^x = e^{x \cdot \ln x}$ that is $\ln u \cdot e^{h \ln u}$. Now it is fairly intuitive that this "instantaneous" rate of change is the slope of the tangent line to the curve at that point (if that's not so intuitive, think of the argument that produced the rate of change result in the first place: we are thinking of the graph $e^{x \ln u}$ as being almost the same as a sequence of very short line segments, with that slope exactly). Now, using the argument that our special value of u will cause the graph of u^x to touch the graph of $\log_u x$ at one point only, lying on the line $y = x$, we can see that this special value of u corresponds to a graph u^x that has $y = x$ as a tangent. No other u will produce this (think of the corresponding graphs - look at the first picture in sec. 3 for help).

Let's write the equation of the tangent line to u^x at a point $\left(h, u^h\right)$ on the graph. By the previous discussion it is a line with slope $u^h = e^{h \ln u}$, going through the point $\left(h, u^h\right)$. Using point-slope form, the line will be

$$y - e^{h \ln u} = \ln u e^{h \ln u} (x - h)$$

which, when turned into slope-intercept form becomes

$$y = x \cdot e^{h \ln u} \ln u - e^{h \ln u} (h \ln u - 1) \tag{1}$$

We can now make good use of what might have seemed an idle observation at the time: *of all the various forms for the equation of a line, slope-intercept is the one for which each line has only one way to be expressed.* This means that if (1) is to be the line $y = x$, we need to have

$$e^{h \ln u} \ln u = 1, \quad e^{h \ln u} (h \ln u - 1) = 0$$

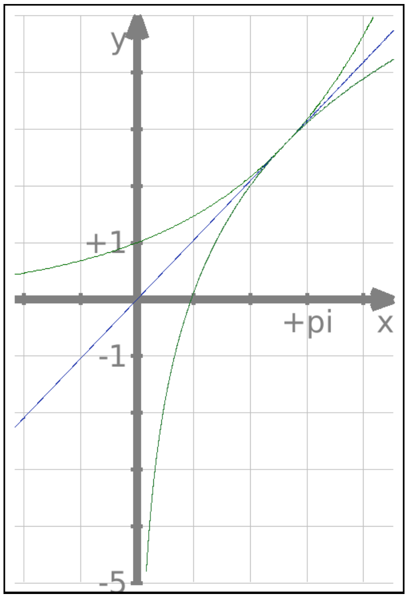
The second equation is easy to solve: since $e^x \neq 0$ for any x, the only way it can work is to have

$$h \ln u = 1$$
$$h = \frac{1}{\ln u}$$

This shows that our line will have the required tangent at the point $\left(\frac{1}{\ln u}, e^{\frac{1}{\ln u} \cdot \ln u}\right) = \left(\frac{1}{\ln u}, e\right)$. But we are also on the line $y = x$, so we can write the point as (e, e), and this means $\frac{1}{\ln u} = e$, or $\ln u = e^{-1}$, or $u = e^{\frac{1}{e}}$! We can confirm this, by taking on the first equation: plug the value of $h = \frac{1}{\ln u}$ in, and get

$$e \cdot \ln u = 1$$
$$\ln u = e^{-1}$$
$$u = e^{\frac{1}{e}}$$

There you have it! This special exponential is $\left(e^{\frac{1}{e}}\right)^x = e^{\frac{x}{e}}$ and, since $\ln\left(e^{\frac{1}{e}}\right) = \frac{1}{e}$, it touches the line $y = x$ (and hence the graph of $\log_{e^{\frac{1}{e}}} x = \frac{\ln x}{\ln\left(e^{\frac{1}{e}}\right)} = \frac{\ln x}{\frac{1}{e}} = e \ln x$) at the point (e, e)!



Looking now at the second graph in sec. 3, you'll notice that this situation never applies when $u < 1$. In fact, in the discussion above, we didn't force $u > 1$ - it just came out that way. So, there's only one base with the feature of having its two graphs "just touching": $e^{\frac{1}{e}} \approx 1.4447$. The "touching" occurs at the point $(e, e) \approx (2.7183, 2.7183)$.

Note: Another way to answer our question would be to look at the intersection of u^x (or of $\log_u x$) with the line $y = x$. This leads in both cases[1] to solving

$$\frac{\ln x}{x} = \ln u$$

If we draw the graph of $\frac{\ln x}{x}$, we can see the general picture. Here is the graph of $y = \frac{\ln x}{x}$[2].

[1] In fact, the system

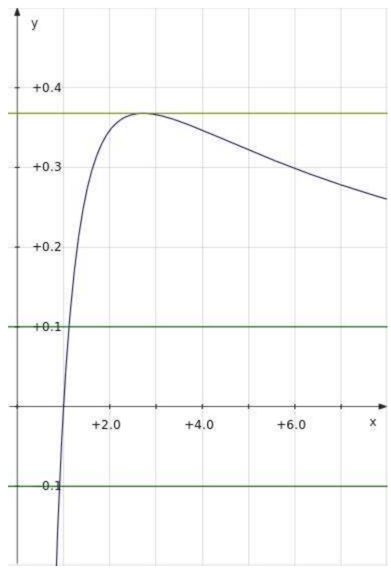
$$\begin{cases} y = & u^x \\ y = & x \end{cases}$$

leads to $x = u^x$, $\ln x = x \cdot \ln u$, and the system

$$\begin{cases} y = & \log_u x \\ y = & x \end{cases}$$

leads to $x = \log_u x = \frac{\ln x}{\ln u}$.

[2] You will notice that the two axes have different units. This is done in order to amplify the variation of the curve in the vertical direction: otherwise it would be difficult to spot the change in behavior as x grows from 0 to 1, to e increasing, and then starts to decrease.



For $y \leq 0$ (i.e., for $0 < x \leq 1$) there is only one solution. Compare the green line $y = -0.1$. Then, for $0 \leq y < \widehat{y}, 1 < x < \widehat{x}$ (right hand values to be determined), we have two solutions: compare the other green line $y = 0.1$. Finally, for $y = \widehat{y}, x = \widehat{x}$ we have one solution - compare the brown line. That's the value we are looking for. This value is also where the function $\frac{\ln x}{x}$ takes its maximum value! Unfortunately, I could not come up with an elementary method to determine this value, but with just minimal calculus it would be easy to realize that $\frac{\ln x}{x}$ reaches a maximum when $\widehat{x} = e$. Consequently, $\ln \widehat{u} = \frac{\ln \hat{x}}{\widehat{x}} = \frac{1}{e}$. Hence, we get the same result as above is started with a question (the intersections of $y = u^x, y = \log_u x, y = x$ where e didn't appear anywhere, or play any special role. And yet, the solution is all about this number! Yet another suggestion that this number has a special standing when dealing with powers and logarithms.

4 Change Of Base Formula In Practice

Here is a tiny program in MATLAB (a C-like language, well known for its robust numerical capabilities):

```
function lg(x,a)
>num=log(x)
>den=log(a)
>lg=num/den
>endfunction
```

This function takes to positive values x, and a, and calculates $\frac{\log x}{\log a} = \log_a x$. MATLAB, like any other computer language, does not have a log function for each possible base. In fact, it only uses natural logarithms (that is, the function `log(x)` returns what our book would call $\ln(x)$). It's easy, however to create your own function that will give you any log in any base, as you can see.