Bases For Logarithms

1 What's a base for a logarithm?

We saw in the book that you can, at least in principle, compute the logarithm of a number in any positive base $a \neq 1$, since $\log_a x$ is simply the inverse function of the exponential function a^x.

However, you will have noticed too that your calculator (and any mathematical software you may meet) lists only logarithm functions with one or two bases. In fact, given the "change of base" formula, if you are able to compute logarithm in a base, you can compute them in any other base.

The bases that are actually used, outside of book problems, are 10, e (a "very" irrational number - strange, but it is the most convenient base of all), and (mostly in specific application fields) 2. Let's see what propels these bases to the forefront.

2 10 as a base for logarithms

You must realize that before the advent of electronic digital computers, calculations had to be performed "by hand". In particular, while adding machines are easy to build, multiplication presents a bigger challenge - not to mention exponentiation. As you might have noted, long calculations involving complicated numbers appear really soon when you start working financial problems (think compound interest). Also, calculations with non trivial numbers come up in all parts of science, as well as in very many applied fields. The way people handled such calculations was to reduce their complexity by referring to a *table of logarithms*.[1]

Suppose you had a book, which (to a certain precision) allowed you to read off the logarithm (in some base a) of all numbers up to really big ones. And suppose the same book allowed you to go back, given the logarithm of a number, and find the number itself. Now, if you had to compute $x \cdot y$, where x and y might be numbers with many digits, you could, first, compute

$$\log_a x + \log_a y = Z$$

[1] For the sake of full disclosure, I have to admit that your instructor knows of this, first-hand. As a high-school student, he was subjected to learning how to use logarithm tables, since, at that time, calculators were not yet available, and computers, with a lot less computing power than your current cell phone, were the size of a big room.

(addition is much easier to do - and an adding machine would speed you up considerably), and then, simply read off which number z it is for which

$$\log_a z = Z$$

(of course, by definition, $a^Z = z$, but that could be a difficult calculation - hence the use of a table).

Once you are set on this program, you will have to choose a standard base a to calculate your table (which is a nontrivial operation, but it has to be done only once, so to speak). It quickly turns out that, since we use a "base 10" numbering system, choosing $a = 10$ makes the preparation of our book considerably simpler[2].

How so? Maybe, it's best to see it through a couple of examples. Suppose we need to compute $\log_{10} 4327, \log_{10} 43.27, \log_{10} 0.4327$. The numbers only differ by multiples of 10... Let's write the first two as

$$4327 = 0.4327 \cdot 10^4 \quad 43.27 = 0.4327 \cdot 10^2$$

Now,

$$\log_{10} 4327 = \log_{10}\left(0.4327 \cdot 10^4\right) = \log_{10}(0.4327) + \log_{10}\left(10^4\right) = 4 + \log_{10}(0.4327)$$

since, clearly, $\log_{10}\left(10^4\right) = 4$. Similarly,

$$\log_{10} 43.27 = \log_{10}\left(0.4327 \cdot 10^2\right) = \log_{10}\left(10^2\right) + \log_{10}(0.4327) = 2 + \log_{10}(0.4327)$$

As you can see, we only need the decimal logarithms of numbers between 0 and 1, to compute immediately the logs of any other number. Actually, with a little manipulation, we can use the decimals logs of numbers between $10^{-1} = 0.1$, and $1 = 10^0$ to compute all other decimal logs. Since we cannot compute the logarithms of "all" numbers between 0.1 and 1, we will limit ourselves to those with a decimal expansion that is less than some reasonable choice (4, 5, 7, ... digits), and, consequently, will be able to compute the logarithm of numbers given with the same *precision*.

This calculation tool influenced the introduction of certain units of measurements that are given in terms of the decimal logarithm of a quantity - like the pH, for measuring the concentration of Hydrogen ions in a solution (its acidity), or decibels, for measuring the intensity of a sound.

Aside from the application we just quoted, you surely realize how the advent of digital calculators and computers have made the use of decimal logarithms all but obsolete! Since the ease with which we can express $\log_{10}$ of a number once

[2] In fact, if Martians existed, and had, say, 12 fingers, they might have chosen 12 as their number base, and the "easy" choice for a logarithm base would have been $a = 12$. As a matter of fact, 10 is a pretty inefficient base for a numbering system - think how most common fractions have an unending decimal expansion, including $\frac{1}{3}, \frac{1}{6}$, etc. - and 12 would have been a better choice (the ancient Babylonians made a gutsy choice - 60 - which produces complicated notation, but very efficient representations for common fractions), but we are so used to counting on our fingers...

we know $\log_{10}$ of the number between 0.1 and 1 we get by multiplying it by a suitable power of 10 is the only good reason to introduce 10 as a base, "common logarithms", as they are also called, have disappeared from the mathematical scene: if you are ever going to take a math class at a level higher than introductory calculus, the notation log, without the mention of a base, will be reversed to *natural* logarithms, and the ln notation will be abandoned (or become very rare) - that's because in calculus you quickly realize that base e is the only base you really want to deal with, and 10 has, by now, no better reason to be chosen as a base than, say, 123, or 43.8901...

3 Base 2

Nobody mentioned base 2 in this course, so why is it interesting? It is interesting nowadays, when we are hooked on *digital* computers. As you know, any object in a computer (numbers, letters, symbols, whatever) is represented as a sequence of 0 and 1. In particular, numbers are expressed in this way by using their representation in base 2. For instance, we write 21 to mean "twenty-one", in that

$$21 = 2 \times 10^1 + 1 \times 10^0$$

The same number can be expressed in terms of powers of 2, as in

$$21 = 16 + 4 + 1 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

as 10101. In general, you can "code" any information as a sequence of 0s and 1s. Now, the *length* of such a sequence is an indication of the relative complexity of the information. How long is the length of the coding for 21? Clearly, 5. What is $\log_2 21$? Well, since $21 = 2^4 + 2^2 + 1$, its $\log_2$ lies between 4 and 5. Hence we can "measure" the "complexity" of a binary coded piece of information by the integer part (plus 1, if you want to be picky) of its $\log_2$.

4 Base e

How in the world did people come upon the idea that e is the "best" base for logarithms (and for exponentials too: instead of computing a^x, people will compute $e^{x \cdot \ln a}$)? The reason will be really clear once you get to calculus. Right now, the convenience of e is far from apparent. We can point to two facts that may provide at least the start for an explanation

4.1 Compound Interest

You know the formula for compound interest: if you are borrowing P dollars at (annual) interest rate r, and the interest is compounded n times in a year (at equally spaced intervals), if you wait t years, your debt will have risen to

$$P\left(1 + \frac{r}{n}\right)^{nt} \tag{1}$$

Now, the late Renaissance was highlighted by the rise of a powerful class of (greedy) bankers, who were quick to note that without touching r, you can squeeze more from your loan, if you increase n. That's how interest grew in finding out what happens to (1) when n increases without bounds.

It is not a trivial fact (which is shown in calculus classes) that the number

$$\left(1+\frac{r}{n}\right)^n$$

is bounded between 2 and 3, and you can conduct your own numerical experiment with your calculator (or computer) to see that, more or less, when n is big enough

$$\left(1+\frac{1}{n}\right)^n \simeq e = 2.71\ldots$$

Thus, the use of the new formula

$$Pe^{rt}$$

for the value of a loan at annual interest rate r, after t years, if interest is compounded "instantaneously". At least, it shows that no matter how often you compound interest, there is a limit on the growth of the debt: a loan at 100% interest rate, will be worth twice its original value after one year, at simple interest; at instantaneously compounded interest, it will be worth more than 2.71 times its original value - a big increase, but not an "infinitely big" one.

4.2 Rate of Growth

This is the most important "good" feature of the function e^{kx} - and, consequently, of $\ln x$. To look at it before calculus, we'll have to be somewhat approximate, but you should get a reasonable idea.

Consider the rate of growth of a compound interest deposit. Let's start at time $t = 0$, and suppose interest is compounded every s years (e.g., $s = \frac{1}{12}$, if interest is compounded monthly). Usually, $s = \frac{1}{n}$, were n is the number of compounding times in a year. Between 0 and s, your money grows linearly: if you start with \$1, and the interest rate is r, you have, after $t < s$, $1 + rt$. At time s, when you are at $1 + rs$, this becomes your new principal, and you start acquiring interest on this sum too. So, for $s < t < 2s$, you will have $(1+rs)+(1+rs)(t-s)$. You see that your money is growing again linearly, but the constant of proportionality, for the time in excess of s, is now your deposit at time s. At $t = 2s$ it all starts over again, and in the next interval your growth will be linear in $(t - 2s)$, with a proportionality constant equal to your total deposit at time $2s$.

Summing up, assuming s is very small, you see that, at any given time, the money in your deposit in increasing, over very small time intervals, proportionally to your total deposit at the most recent time equal to a multiple of s. Hence, as s becomes very small, (if $s = \frac{1}{n}$, as n becomes very big), you could say that your deposit is growing, at each instant, at a rate proportional to its current size (after all, you are always at a very short time from the last ,multiple of s). This property is a property of the function e^{kx}, of course: approximately, over a time interval $[ms, ms+t]$, it will grow at a rate equal to $ke^{k(ms)}$. Since t cannot be too big (it is less than the very small s), we can say that the rate of increase of the function e^{kx} is practically equal to ke^{kx} ($x = ms + t \approx ms$).

This "self-proportional" rate of growth is true for every exponential function a^{kx} ($a > 1$), but the proportionality constant happens then not to be k, but $k \cdot \ln a$. Thus the choice of e as a base, simplifies the analysis of how your function is growing!

All of this can be made much more precise and clear with the tools of calculus (which, in fact, was developed specifically to handle things like rate of change in functions). The bottom line, though, is that exponential functions have this property (and they are the only ones that do) of "self-proportional growth", that suggests their use in any situation in which you can assume that rate. Additionally, among all possible expressions for an exponential functions, the one using e as a base is the clearest, since the proportionality constant is precisely the coefficient in the exponent!

One of the most famous examples is the Malthusian model of population growth: assuming that a population growth rate has to be, at each time, proportional to its size (the more individuals, the more offsprings), Malthus came up with the exponential growth model $P(t) = P_0e^{kt}$, where P_0 is the population at time $t = 0$, and k is the proportionality constant measuring the "fertility" of the population.

Negative growth rates mean, of course, rapid decline, and a prototypical application is the standard model for radioactive decay: starting at time $t = 0$ with a certain mass M_0 of radioactive material, its decay over time will leave us, at time t, with a mass of $M(t) = M_0e^{-kt}$, where $k > 0$ is a constant measuring the rate of decay (this is a rate of decrease for the mass: the more mass, the more chances for an atom to decay, hence a faster rate of decline for the function $M(t)$).