Logarithmic Functions

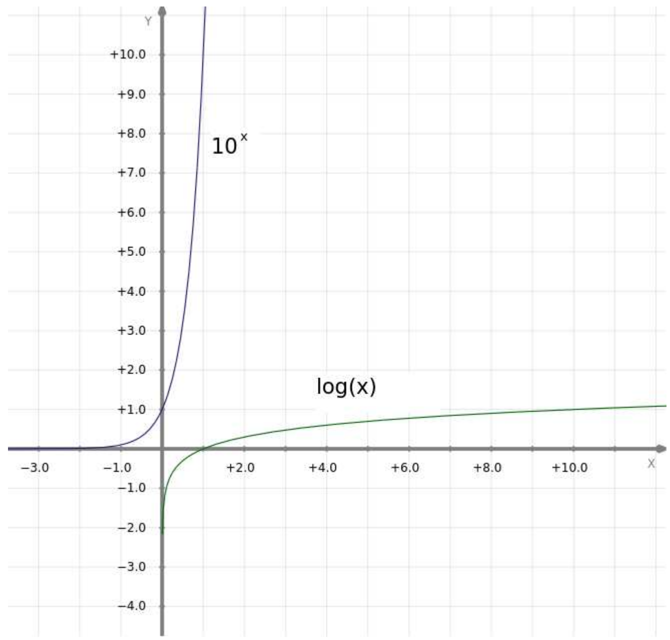
Looking at the graph of an exponential function we will notice that no two different values of the input will produce the same output. Functions with this property are called *one-to-one* (one input to one output), and are *invertible*, that is, we can find one input only that will produce a given output. This "reversed" function (we choose the output and find the corresponding input) is called the *inverse function* of our original function. If the original function was called f, the inverse is denoted by f^{-1}

Note This is *not* the same as $\frac{1}{f}$ (the reciprocal function)! It is maybe unfortunate that this notation is standard but possibly causing some confusion, since the reciprocal of a number a is denoted by both $\frac{1}{a}$ and a^{-1}).

Since exponential functions of the form $f(x) = b^x$ are one-to-one, we can define their inverse functions, called *logarithms*[1] $f^{-1}(x) = \log_b(x)$. By definition, if $y = \log_b(x)$, then $b^y = x$, and if $b^x = y$, then $x = \log_b(x)$. You will often see the shorthand notation $\log_b x$ for $\log_b(x)$. b is called the *base* of the logarithmic function $\log_b(x)$. Note that, as $\log_b(x)$ is the inverse function of b^x, b^x is the inverse function of $\log_b(x)$. This is a general fact: $\left(f^{-1}\right)^{-1} = f$!

Since b^x is only defined for $b > 0$, we can only consider positive number as bases. Also, since $b^x > 0$ for any x, we can only consider logarithms of positive numbers. On the other hand, logarithms can take any real value (check their graph if you don't find this statement obvious).

[1] The exotic name comes from the Greek language, as John Napier, who introduced the notion, combined the Greek words for "ratio" and "number".



The graphs are for the exponential function 10^x and its inverse, $\log(x)$ (when no base is specified, log stands often for $\log_{10}$)

Properties of logarithmic functions follow from the corresponding properties of their inverse functions, exponential functions:

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
- $\log_b\left(x^k\right) = k \cdot \log_b(x)$
- $\log_b(1) = 0$
- $\log_b(b) = 1$
- $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$

for any positive number b and any positive numbers x and y.

Remark In practice, only two bases are used (that's why you only have two log keys in your calculator). The choice $b = 10$ is common in science, where logarithms of constants or measured quantities make it is easy to transition between logarithms and scientific notation. When considering exponential functions (e.g. of time), it turns out that the natural base is an irrational number denoted by e. You will consider this case if you take a Precalculus and a Calculus class.

Note In computer science and information science it is also common to use 2 as a base for exponentials and logarithms.