Review of Linear Functions

1 Lines

This material is a review of familiar results. We only want to stress the following fact. A line can be represented algebraically in many equivalent ways.

1. We may look at the *slope* of the line, and its *intercept* with the $y-$axis. We use these numbers to write the *slope-intercept* form. For example,
$$y = \frac{2}{3}x - \frac{3}{4} \tag{1}$$
has slope $\frac{2}{3}$, and intercepts the $y-$axis at the point $\left(0, -\frac{3}{4}\right)$. Note that, in this context, "slope" refers to the increase in the y variable, per unit increase of the x variable.

2. We may look at the line with given slope, and going through a point. This generalizes case 1, where the point was the (unique) point where the line crosses the $y-$axis, while now it can be any point. So, for instance, the same line as (1) can be represented by
$$\left(y - \frac{1}{4}\right) = \frac{2}{3}\left(x - \frac{3}{2}\right) \tag{2}$$
called *point-slope* form (slope is still $\frac{2}{3}$, and now we refer to the point $\left(\frac{3}{2}, \frac{1}{4}\right)$, which also belongs to the line – note that, subtracting $-\frac{1}{4}$ from both sides of (2), and opening the parenthesis, we revert to (1).

3. With a little algebra, we can write an equation for the same line that puts the two variables on equal footing (they are treated differently in the first two forms). Indeed, we can rewrite (1) in successive steps as
$$y - \frac{2}{3}x = -\frac{3}{4} \tag{3}$$
$$3y - 2x = -\frac{9}{4} \tag{4}$$
$$12y - 8x = -9 \tag{5}$$
Please, note that all three equations all represent the same line as (1), and (2). The book calls (5) *standard form.* Another standard form for the same line could be
$$24y - 16x = -18 \tag{6}$$
and so on.

4. Another "symmetric" form for a line (one that is not mentioned as such in the book) is the form specified by two points through which the line has to go. If the points are (x_1, y_1), and (x_2, y_2), an equation can be written as
$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \tag{7}$$
(pay attention to the order in which the coordinates are used). For example, our friendly line can be also specified by the fact that it goes through $\left(0, -\frac{3}{4}\right)$, and $\left(\frac{3}{2}, \frac{1}{4}\right)$, giving
$$\frac{y + \frac{3}{4}}{\frac{1}{4} + \frac{3}{4}} = \frac{x}{\frac{3}{2}} \tag{8}$$
With minimal algebra, form (8) can be turned into any of the others, of course.

5. Finally, it is obvious that there are also infinitely many ways to choose our points and write an equation in a form like (7). Note also, that, given the denominators in (7), this is not a way to handle lines of the form $x = c$, or $y = c$ (i.e., vertical or horizontal lines), since one of the two denominators would be then zero.

Short Observations

There are a few things to note for each form.

1. Slope-intercept form (1) explicitly shows two specific characteristics of the line. Two different lines will differ in at least one of them, so there is only one slope-intercept form for a line. *There are lines that cannot be represented in slope-intercept form*: lines of the form $x = c$, where c is some number, have no slope, and they coincide with the $y-$axis, if $c = 0$, and never intercept it if $c \neq 0$.

2. A given line can be represented in many ways in point-slope form: the slope is a fixed number, but the point can be any of the infinitely many points on the line. For example, instead of $\left(\frac{3}{2}, \frac{1}{4}\right)$, or $\left(0, -\frac{3}{4}\right)$, we could, just as well, use $\left(\frac{9}{8}, 0\right)$, or $\left(1, -\frac{1}{12}\right)$, and so on. Again, lines with no slope, i.e., lines like $x = c$ (vertical lines), cannot be represented this way.

3. Note that in the standard form $Ax + By = C$ none of the constants in any of the forms need to be an integer, or even a rational number (a fraction) for the equation to represent a line. For example,
$$\sqrt{2}y - \pi x = 1$$
is a perfectly legitimate line. Please, note also that *any line* can be represented in standard form. On the other hand, as illustrated in equations (3) - (6), there are infinitely many standard forms for a given line.

Application

Here's an application we'll meet in some word problems: a business has to set the price of a product, and it will sell more of it, the lower the price - let's model the relation between price and sales. For simplicity, we will assume that the relation is linear. That is, if s is the sales number, and p is the price, the pairs (p, s) in the plane lie on a line. Depending on what data we are starting from, we may use any of the forms to specify the line.

1. If we are told, e.g., that for every dollar we raise the price, we will sell 10 less items, and that if we just gave the stuff away for free, we would "sell" (so to speak) 1,000 if them, then we know that the slope of the line is -10, and that the $s-$intercept (i.e., the $y-$intercept in our discussion above, corresponding to $p = 0$) is at $(0, 1000)$. Hence, the equation is

$$s = -10p + 1000$$

2. On the other hand, if we are told the slope, as before, but also that a price of \$50 will produce 500 sales, we would write

$$s - 500 = -10\,(p - 50)$$

3. We might want to represent the relation in a more symmetric form, since we might, hypothetically, want to set a sales goal, and set the price accordingly, as well as going the reverse route. Hence, we could write our line in the form

$$s + 10p = 1000$$

and, if we had the goal of writing price as a function of sales, "solve for p", and end up with

$$p = 100 - \frac{s}{10}$$

or any of the possible variations.

4. Finally, if told that, for example, we would give away 1,000 items if we gave them away, but would stop selling any at all, once we raised the price to \$100, we would know that our line goes through the points $(0, 1000)$, and $(100, 0)$, and can then be written as

$$\frac{s - 1000}{0 - 1000} = \frac{p - 0}{100 - 0}$$

2 Linear Functions

These are the ones appearing in what is called the "slope-intercept" form for the equation of a line. All have the form

$$f(x) = ax + b$$

where a and b can be any real number. This excludes vertical lines, which, as you may observe, are not functions.

Linear functions are the polynomials of degree one. Like all polynomials, the domain of all linear functions is $(-\infty, \infty)$, that is "all real numbers", or $\mathbb{R}$ (all equivalent notations/expressions).

Useful facts you may want to keep in mind are

1. f is increasing if $a > 0$, is constant if $a = 0$, is decreasing if $a < 0$
2. The average rate of change is a - it's not average at all, in that it is constant, and always the same, no matter which two points you choose for its evaluation: it is the rate of change of f.
3. A non constant linear function *always has exactly one zero*. Try to make a complete argument for this statement. Clearly a constant function (which is automatically linear, $a = 0$) has either no zeros at all (if $b \neq 0$), or all the domain consists of zeros (if $b = 0$).

3 Some Additional Comments

Any function that is not linear is called *nonlinear*. We have been already working with polynomials, which are nonlinear, as long as their degree is greater than 1. Notice, that, when x is very close to 0, i.e. for $|x|$ small enough, the graph of a nonlinear polynomial $f(x)$ will be very close to the graph of the linear function you obtain if you ignore all terms of degree higher than one. This linear function is called the *tangent to the graph* at the point $(0, f(0))$. By clever horizontal shifting, we can find the tangent to the graph of a polynomial at *any* point $(x, f(x))$. There is a more detailed discussion in another additional material file.

A standard use of the existence of a tangent at a point (which can be suitably extended to a much wider class of functions, besides polynomials) is to approximate the "true" function with a linear one. That's how many of the linear models we encounter in science originate.

3.1 Example: Hooke's Law and Ohm's Law

If we take a spring (or any elastic material) that has length l when it is neither pushed nor pulled, if we apply a force F pulling or pushing the spring, the new length will be of the form

$$L = l + kF$$

where k is a constant depending on the material, and the geometry of the spring, and F is positive if it is pulling the spring. We have here L a linear function of F

This can be viewed as an experimental law, or argued theoretically by assuming that the (unknown) "true" law $L = f(F)$ is of polynomial type (or that it is as a more general function, that still has a tangent line at $L = l, F = 0$).

The same logic explains *Ohm's Law* in electricity, where the intensity of a current I through a circuit is a function of the voltage V applied to the circuit according to the function

$$I = \frac{1}{R}V$$

(when $V = 0$, there's no current, and so $I = 0$). The slope $\frac{1}{R}$ is called the *conductance* of the circuit, and its reciprocal, R, is called the *resistance* of the circuit.

4 Interpolating Linear Functions To Empirical Data

This topic is one of the most common applications of statistics (the method actually can be used to interpolate many other types of functions, but the linear case is the most common). Suppose you have a *scatterplot* and wish to write an equation for the line that "best approximates" the plot. Notice that we need a definition of "best approximation" here. Also, the plot need not look like it is clustering around a straight line for the method to work! In fact, sometimes linear approximations are made to data that looks very nonlinear. As in all things, it all depends on what you are doing this for.

The standard choice for "best approximation" (which can be justified in different ways, but, of course, still has some arbitrariness in it), is to choose the line $ax + b$ such that, if the points in the scatterplot are listed as $(x_i, n_i); i = 1, 2, \ldots n$ assures that the difference

$$(y_1 - (ax_1 + b))^2 + (y_2 - (ax_2 + b))^2 + \ldots + (y_n - (ax_n + b))^2$$

is as small as possible. Since we are trying to get to the smallest value of a sum of squares, this is called the *Least Mean Square* criterion. Since we need to choose *two* values, a and b, this requires handling two different variables and find the couple that causes this expression to be as small as possible. We would need to work a bit more than we are expected to in this quarter in order to find the solutions to this problem, but let us just say that it is not hard (given the appropriate tools) to see that there is exactly one best choice for these two numbers, and this is what your calculator (if it has this function pre-programmed) will tell you if you ask it to give you the "best fit". If you are still curious, you can check the file on *Least Mean Squares.*

Of course, you can always pick two points among the data and draw the line between the two. If you were careful enough, the line might be a reasonable "best fit" - but only in an intuitive sense: somebody else's choice will more than likely be different. As long as you are only looking for a rough idea, this is OK. You would not use this eyeballing method in a rigorous context, though.