Domains, Ranges, and Inverse Functions

Let's go over how the concepts in the title interplay when defining inverse functions. First, here is the situation in general.

Domain and range of inverse functions

Let f be a function, defined on some domain D, with range E (in math, this is shortened to $f : D \to E$), that is for all x in the set D, we have $f(x)$, a number in the set E, and for every y in the set E, there is (at least) an x in the set D, such that $f(x) = y$. In math this long phrase is shortened to

$$\forall x \in D \Rightarrow f(x) \in E \text{ and } \forall y \in E \Rightarrow \exists x \in D : f(x) = y$$

(These shorthand might look like hieroglyphs at first, but, once you get used to them, they make it very easy to grasp the message very rapidly).

Now, assume that f is one-to-one on D, that is, if $x_1 \neq x_2$, both in D, $f(x_1) \neq f(x_2)$. This allows us to define the inverse function to f, f^{-1}, whose domain is E, and whose range is D, by $f^{-1}(y) = x$, if (and only if) $f(x) = y$ (shortened in math as $f^{-1}(y) = x \Longleftrightarrow f(x) = y$. This implies that $\left(f^{-1}\right)^{-1}$ has domain D and range E, and that $\left(f^{-1}\right)^{-1}(x) = f^{-1}(y) = x$, so $\left(f^{-1}\right)^{-1} = f$. Also, we see that $f \circ f^{-1}$ has domain E, and range E as well, since $f^{-1}(y)$ is in D, and f maps D to E, and $f \circ f^{-1}(y) = f(x) = y$. So $f \circ f^{-1}$is the *identity map* on E (the identity map is the map from a set to itself, which maps a number to itself: $i : F \to F$, $\forall z \in F, i(z) = z$, using the shorthand above). Similarly, you can check that $f^{-1} \circ f$ is the identity map from D to D.

Let's see how this translates in a few simple examples.

1 $f(x) = \frac{1}{x}$

Clearly, here the domain is $D = (-\infty, 0) \cup (0, \infty)$, and so is the range, because the equation

$$y = \frac{1}{x} \tag{1}$$

has always a solution, $x = \frac{1}{y}$, except if $y = 0$ (if it was possible that $\frac{1}{x} = 0$, then, multiplying by x both sides, we would end up with the contradiction $1 = 0$!). Thus $E = D$, and we also found, since there was only one solution to (1), that the function is one-to-one on D, and that $f^{-1}(x) = \frac{1}{x}$.

Now, following the argument in sec , we have that $\left(f^{-1}\right)^{-1}$ has domain and range equal to $D=E=(-\infty,0)\cup(0,\infty)$, and that on this domain,

$$f\circ f^{-1}(x)=f^{-1}\circ f(x)=x \tag{2}$$

.

Note that the equations in (2) **hold only for x in D, that is for $x\neq 0$!** They don't apply if $x=0$, because if you try to plug in 0, for example, into $f^{-1}\circ f$, you have to evaluate first $f(0)$, and that is illegal, since $f(0)$ is not defined. If you think this is nitpicking, think of a computer, where you have coded two routines, call them `function f()` and `function finv().` If you call `f()` with argument 0, your computer will respond with an error message: `DIVISION BY ZERO,` and stop. Now, if you write the instruction `finv(f(0))`, your computer will first try to evaluate `f(0)`, and will stop right away with the previous error message. The same would happen if you gave the instruction `f(finv(0))`, of course.

Don't get fooled by the apparent algebra

$$f^{-1}\circ f(x)=f^{-1}\left(\frac{1}{x}\right)=\frac{1}{\frac{1}{x}}=1\cdot\frac{x}{1}=x$$

This works only when $x\neq 0$, because $\frac{1}{\frac{1}{x}}$ makes no sense if $x=0$. You always need to check the domain and range of f when working examples like this.

2 $f(x)=\sqrt{x}$

This function has domain $D=[0,\infty)$, and range $E=D$. To find whether it is one-to-one, we solve the equation $y=\sqrt{x}$ for x. Note that $\sqrt{x}$ is the symbol for the *principal square root*, that is the non negative square root (also written as $x^{1/2}$). Here, $y\geq 0$, since it has to be in $E=D$. Squaring, we get $x=y^2$, so f^{-1} is a square function, but not the ordinary square function! It is the square function with domain $[0,\infty)$! That said, you can check that all the previous calculations apply here as well, *as long as we work with non negative numbers.*

What if, instead, we worked with $g(x)=x^2$? This function is not one-to-one, since $g(-x)=g(x)$, hence it has no inverse[1]. Inspired by the previous paragraph, we can define a new function $h(x)=x^2$, with domain $[0,\infty)$ and the range turns out to be the same as the domain again. This is one-to-one, and it is easy to check that it has inverse $h^{-1}=f$, where f is the square root function defined in the first paragraph.

We can also define another one-to-one modification of g, by defining $k(x)=x^2$, with domain $(-\infty,0]$. The range is now different from the domain: it is $[0,\infty)$. This too is one-to-one, and it is has an inverse function, with domain the range of k, $[0,\infty)$, and range the domain of k, $(-\infty,0]$: it is easy to see that $k^{-1}(x)=-\sqrt{x}$, the *negative square root of x.*

[1] Here is a classic "trap" question: evaluate $\sqrt{(x^2)}$. If you answered x, you were wrong. If $x<0$, you get $-x$, so the correct answer is $\sqrt{(x^2)}=|x|$.

Finally, we can explore combining f and g. It should be clear that $f \circ g$ has domain $(-\infty, \infty)$,since that's the domain of g,which has range $[0, \infty)$, hence $f \circ g$ is always defined. On the other hand, $g \circ f$ has domain $[0, \infty)$ (and, of course, the same range). Again, don't be fooled by the apparent algebra

$$\left(\sqrt{x}\right)^2 = x$$

because that's true only if $x \geq 0$ (think of our computer again), so it is not true for negative x (the combination is undefined there).

3 $f(x) = \frac{ax+b}{cx+d}$

In the formula above, we assume that a, b, c, d are any non zero real numbers. Again, we check whether this is one-to-one by solving $y = \frac{ax+b}{cx+d}$ for x. We have

$$y(cx + d) = ax + b$$

$$cyx - ax = b - dy$$

$$x\left(cy - a\right) = b - dy$$

$$x = \frac{b - dy}{cy - a}$$

Of course, this is valid as long as $x \neq -\frac{d}{c}$ (that defines the domain of f), and $y \neq \frac{a}{c}$ (that defines the range of f). Again, this shows that f is one-to-one on its domain, and that it has as inverse the function $f^{-1}(x) = \frac{b-dx}{cx-a}$. Now, it is a little algebra gymnastics to verify that, as it should, $f \circ f^{-1}(y) = y$, and $f^{-1} \circ f(x) = x$:

$$f \circ f^{-1}(y) = f\left(\frac{b - dy}{cy - a}\right) = \frac{a\left(\frac{b-dy}{cy-a}\right) + b}{c\left(\frac{b-dy}{cy-a}\right) + d} = \frac{\frac{ab-ady+bcy-ab}{cy-a}}{\frac{bc-cdy+dcy-ad}{cy-a}} =$$

$$= \frac{ab - ady + bcy - ab}{bc - cdy + dcy - ad} = \frac{bcy - ady}{bc - ad} = y$$

Similarly, you can get some exercise and check that $f^{-1} \circ f(x) = x$ – both equality only holding over the respective domains, that is $y \neq \frac{a}{c}$, and $x \neq -\frac{d}{c}$, respectively.

For example, let $a = 2, b = 3, c = -4, d = 1$. Solving $y = \frac{2x+3}{1-4x}$ for x, we get, indeed

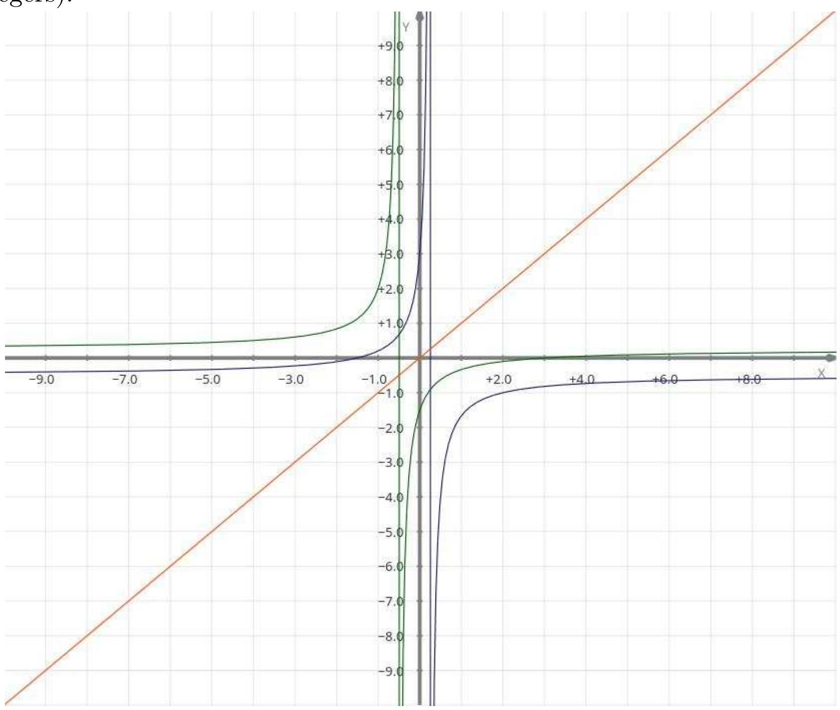
$$y - 4xy = 2x + 3$$

$$2x + 4xy = y - 3$$

$$x\left(2 + 4y\right) = y - 3$$

$$x = \frac{y - 3}{2 + 4y}$$

So $f^{-1}(x) = \frac{x-3}{4x+2}$. With a little patience, $f \circ f^{-1}(x) = \frac{2\cdot\left(\frac{x-3}{4x+2}\right)+3}{1-4\left(\frac{x-3}{4x+2}\right)} = \frac{\frac{2x-6+12x+6}{4x+2}}{\frac{4x+2-4x+12}{4x+2}} = \frac{2x+12x}{2+12} = x$. You can try any other combination (and the constants need not be integers).



The graphs correspond to f (green), f^{-1} (blue), and $f \circ f^{-1}$ (red). The latter has "holes" at $-\frac{1}{2}$, and at x, such that $\frac{x-3}{4x+2} = \frac{1}{4}$ (when $f^{-1}(x)$ falls outside the domain of f), that is$x - 3 = x + \frac{1}{2}$, which is impossible: not surprisingly, the range of f^{-1} coincides with the domain of f, so it cannot map a number into a number where f is not defined.