A Note on Inverse Functions

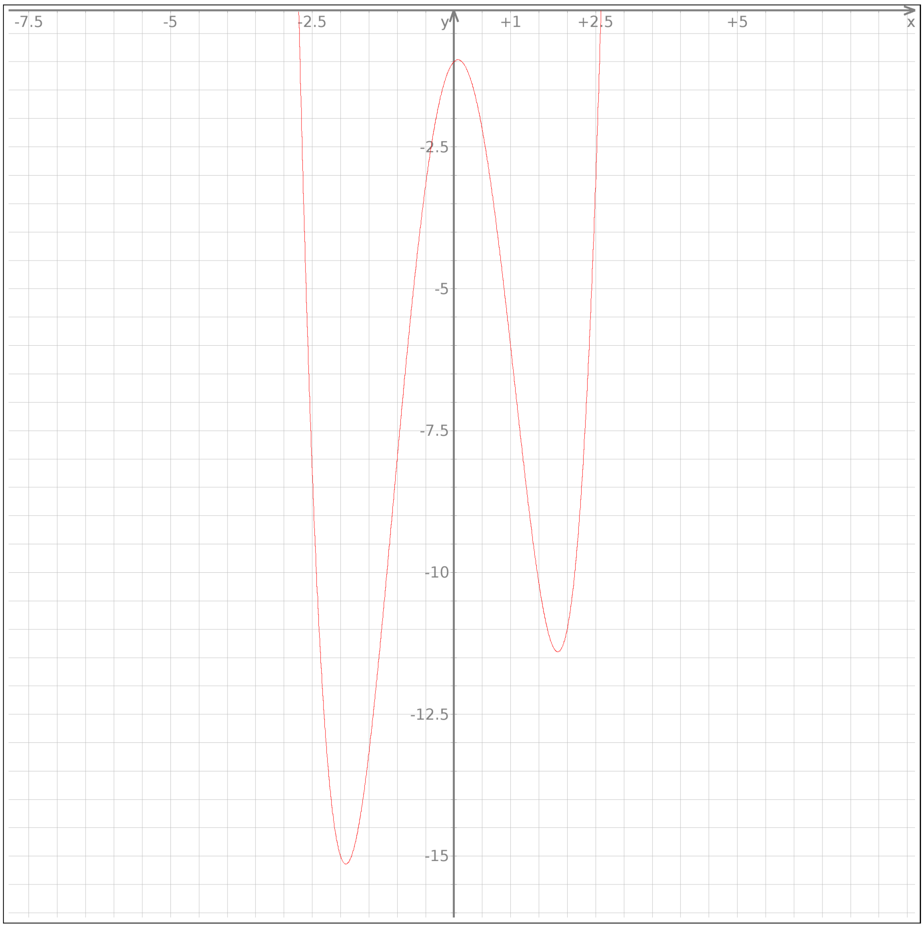
We should take notice of a detail when checking for the existence of an inverse for a give function. The typical example is the function $y = f(x) = x^2$. Clearly, this function has no inverse - for example, it fails the "horizontal line test".

However, a function is not only a "rule", but also includes a "domain". When defining $f(x)$ without mentioning a domain explicitly, like we did above, we imply that the domain is given by all x such that $f(x)$ makes sense. In the case of x^2 this is "all real numbers", of course. And, as we already mentioned, this function has no inverse.

However, we are free to define a *new* function, by specifying the same "rule", but a different domain. The standard example is to take this new function g as $g(x) = x^2$, but domain limited to $\{x\,|x \geq 0\}$. This "new" function does indeed pass the horizontal line test! In fact, this function is invertible, and you have met the inverse function for sure: it is a function with the same domain as g, and is $g^{-1}(x) = \sqrt{x}$ ($\sqrt{x}$ is defined as the *positive* or *principal* square root of x).

Of course, once we feel free to choose a domain with some arbitrariness, we have more than one option. One other obvious choice, in our example, is to restrict to the domain $\{x\,|x \leq 0\}$. This new function, different both from f, and from g, let's call it h, also passes the horizontal line test, and so is invertible. The inverse, is also most likely familiar to you: $h^{-1}(x) = -\sqrt{x}$ (the *negative* square root).

x^2 is by far not the only example. Functions of the form x^{2k} (i.e., even powers) have exactly the same feature. Polynomials present a more complicated picture, but, if you look at the graph of one, if it doesn't pass the horizontal line test, you can notice that, if you choose a sufficiently reduced domain, you can define (in many different ways) a new function that is invertible.



You may notice that this graph (it is the graph of $f(x) = x^4 - 7x^2 + x - 1$) fails the horizontal line test miserably. However, the graph can broken down into four parts, in each of which it is always increasing, or always decreasing. If we define a new function $g(x)$ with the same algebraic expression as f, *but with a domain (arbitrarily) restricted to one of these four sections*, g will be invertible.

In future Math classes you may meet other "standard" functions with a similar situation: while they do not have an inverse, if defined with their "maximal" domain, useful inverses are nonetheless introduced by restricting the original function to a smaller domain.