Trigonometric Identities

The identities listed here refer to trigonometric *functions*. That is, they *do not include any triangle-related identity* (like the Law of Sines, and such).

There is no way we could list all possible identities. In a way, they are endless. The following list is a selection that covers most common identities. More can be derived from these, of course.

1 Fundamental Identities

1.1 The really basic ones:

1. $\tan x = \frac{\sin x}{\cos x}$
2. $\sin^2 x + \cos^2 x = 1$

1.2 Auxiliary functions:

1. $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\tan x}$
2. $\sec x = \frac{1}{\cos x}$
3. $\csc x = \frac{1}{\sin x}$

Cotangent is sometimes really used for brevity. Secant and cosecant are not really in common use, except as trivial shorthand for their expression.

1.3 Identities connecting shifted angles

1. $\cos x = \sin\left(\frac{\pi}{2} - x\right) = \sin\left(x + \frac{\pi}{2}\right) = -\sin\left(x - \frac{\pi}{2}\right)$
2. $\sin x = \cos\left(\frac{\pi}{2} - x\right) = \cos\left(x - \frac{\pi}{2}\right) = -\cos\left(x + \frac{\pi}{2}\right)$
   (these automatically imply similar relations for secant and cosecant)
3. $\cot x = \tan\left(\frac{\pi}{2} - x\right)$
4. $\tan x = \cot\left(\frac{\pi}{2} - x\right)$

More similar identities can be obtained using the sum/subtraction formulas listed below.

1.4 Periodicity:

1. $\sin(x + 2\pi) = \sin x$
2. $\cos(x + 2\pi) = \cos x$
3. $\tan(x + \pi) = \tan x$

1.5 Odd/even properties

1. $\sin(-x) = -\sin x$
2. $\cos(-x) = \cos x$
3. $\tan(-x) = -\tan x$

1.6 Addition formulas

1. $\sin(x + y) = \sin x \cos y + \sin y \cos x$
2. $\cos(x + y) = \cos x \cos y - \sin x \sin y$

1.7 From which you get the double formulas:

1. $\sin 2x = 2 \sin x \cos x$
2. $\cos 2x = \cos^2 x - \sin^2 x$

2 Important derived identities

A commonly used formula is

$$\sec^2 x = \frac{1}{\cos^2 x} = 1 + \tan^2 x$$

From the addition formulas and the odd/even properties it is easy to see that

1. $\sin(x - y) = \sin x \cos y - \sin y \cos x$
2. $\cos(x - y) = \cos x \cos y + \sin x \sin y$

Now a lot of identities follow. For example

1. $\sin(\pi - x) = \sin x$
2. $\cos(\pi - x) = -\cos x$
3. $\tan(\pi - x) = -\tan x$

A little algebra, and the definition of tangent, will yield

1. $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
2. $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$
3. $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

2.1 Half-angle Formulas

A first type is the following. Note how they are ambiguous, as you need to know something more (the quadrant where $\frac{x}{2}$ falls) in order to choose the sign.

1. $\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$
2. $\cos \frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$

The tangent formula is nicer:
$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$

2.2 Reverse Half-angle Formulas

The reverse formulas are actually really useful in some specific calculus operations:

1. $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$
2. $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$
3. $\tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}$

3 Less well-known, but sometimes useful identities

First, from the doubling and sum formulas, it is easy to derive formulas for triple, quadruple, and so on angles. For example:

1. $\sin 3x = 3\sin x - 4\sin^3 x$
2. $\cos 3x = 4\cos^3 x - 3\cos x$
3. $\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

Another group of identities is obtained by reading the sum/subtraction formulas backwards. Although they are less common, it is really good to know they are there, so you can refer to them when in need (and you might be surprised at how often this might happen).

3.1 "Product-Sum"

1. $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$
2. $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$
3. $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$
4. $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

3.2 "Product"

1. $\sin x \cos y = \frac{\sin(x+y)+\sin(x-y)}{2}$
2. $\cos x \cos y = \frac{\cos(x+y)-\cos(x-y)}{2}$
3. $\sin x \sin y = \frac{\cos(x-y)-\cos(x+y)}{2}$