Functions

1 Relations and Functions

A "relation" is a set of ordered pairs. This is a bit abstract, so we'll just say that it is a "rule" which allows us to determine whether two quantities (we'll only work with numbers here, but, in principle, it could be almost anything) are connected by the relation or not. In this class we are mostly interested in mathematical relations, expressed by a formula - usually one or more equations (but they could be inequalities too): when the two numbers are plugged in the two variables in the formula, and the formula is true, the two numbers are connected, otherwise they are not.

Examples are

$$x^2 + y^2 = 1 \tag{1}$$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ are connected, whereas, say, $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$, are not. You can work out examples and counterexamples for the following relations.

$$\frac{x^3 - 1}{1 - xy} = x^2 y^2 \tag{2}$$

$$x = 4y \tag{3}$$

$$y = 4x \tag{4}$$

$$s = 4t^2 - 5t + 1 \tag{5}$$

$$u^2 - v^2 = -1 \tag{6}$$

etc. These are relations too: $x^2 + y^2 \leq 1$, $y > 3x^3 - 2x + 1$, $x + y < 1$, ...

When there is only one value of one of the two variables (say, it is the variable s), that is connected to each value of the other variable (say, it is the variable t), we say that the relation is *a function, from the first variable to the second,* or that *the second variable is a function of the first.* With the variables we just mentioned, we would write $s = f(t)$, or $s = s(t)$, or variants of this notation. For instance, (1), (2), and (6) are not functions. (3) can be viewed as a function from x to y, or as a function from y to x. The same holds for (4). (5) is a function form t to s, but not from s to t.

2 Domain and Range

We can talk of the domain of a relation, but, for simplicity, we'll consider now only functions[1]. Suppose we have a formula, defining a function from, say, x to, say, y. Symbolically, we'll shorten this by writing $y = f(x)$, $f(x)$ is supposed to be an explicit rule: plugging in a number x will produce a unique output y. However, since not all operations can be applied to all numbers, there may be x's that we cannot plug in the formula, because the formula doesn't make sense when that input is used. We may also want to restrict the set of allowable numbers for inputs. The domain of a function is the set of numbers for which it is possible to calculate the corresponding y, whether for mathematical reasons, or because we have to restrict the inputs somehow. This set may be mentioned explicitly, when we wish/have to consider the formula only for certain values of the variable. For instance, we could be describing the motion of a car traveling for a trip on a road, and write it as a function of the distance traveled, s, as a function of time, t. However, we measure time form the moment the car starts, calling that $t = 0$. Suppose the car moves at the constant speed of 50 mph. Let's measure s in miles, and t in hours: the function is $s = s(t) = 50t$. The function $s(t)$ is now defined only for $t \geq 0$, since know nothing about what happened before. Also, the car will stop after some time, say after 5 hours and 30 minutes. Since it makes no sense to consider times greater than that, the complete domain of our function is $\{t\,|0 \leq t \leq 5.5\}$, or $[0, 5.5]$.

If no explicit mention of the domain is provided, it is understood that the domain is the largest possible set for which the expression makes sense. Operations that do not always make sense are, among others,

- divisions (you cannot divide by zero)
- even roots of negative numbers (no real number raised to an even power will result in a negative number)
- More constrained functions will be introduced in your future math classes

For example, look at the following functions where the "critical" operation is a division. If we consider a function like $\frac{x+1}{x-1}$, this is very nice, unless we try to set $x = 1$, which would result in a division by zero. Hence, its domain is $\{x\,|x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$. On the other hand, for a function like $\frac{4x-5}{2}$ there is no problem: the denominator is never zero. Similarly for the function $\frac{2}{1+x^2}$, since $x^2 \geq 0$ can never be equal to -1.

[1] There are a couple of reasons for this. First, and foremost, we (and most of mathematics) will deal mainly with functions. Also, in a relation, the distinction between the two is essentially conventional. Even though we want to distinguish between a "first" and a "second" element in a relation that is not a function, they essentially play a perfectly symmetrical role, at least mathematically (in applications, we still might want to distinguish between "input" and "output", even if the output is not unique). On the contrary, in a function the distinction between "input" and "output" is substantial.

A function may have several "delicate spots", in which case we need to account for all possible failures. For example, look at

$$\frac{\sqrt{2x+1}}{x-2} - \frac{\sqrt{2x-1}}{x+1}$$

Here we need to make sure that *all* operations make sense. Checking each issue, one by one, we have:

- We need to have $2x+1 \geq 0$, for the first square root to make sense. It is easy to check that this requires $x \geq -\frac{1}{2}$
- We need to have $2x-1 \geq 0$, for the second square root to make sense. It is easy to check that this requires $x \geq \frac{1}{2}$
- We need $x-2 \neq 0$, for the first denominator to be non zero. This means $x \neq 2$
- We need $x+1 \neq 0$, for the second denominator to be non zero. This means $x \neq -1$

Now we must combine all these requirements. First of all, we see that the second requirement $x \geq \frac{1}{2}$ makes the first and last superfluous: if $x \geq \frac{1}{2}$, certainly x cannot be -1, and is automatically greater than $-\frac{1}{2}$. We must still impose $x \neq 2$, so that we end up with the following domain:

$$\left[\frac{1}{2}, 2\right) \cup (2, \infty)$$

3 Inverse Functions

This is a topic that we will discuss in more detail later (see the files on inverse functions). However, it is useful to start thinking about it already. Consider a function: this means we have a "rule" that assigns a single "output" to any admissible "input". However, there is no reason why two different inputs might not have the same output.

Think of salaries in a business as the output, and employees as the input. Nothing prevents two employees to have the same salary - but each employee has a uniquely defined salary. Hence we have a function SALARY(EMPLOYEE). If we were looking at the database of this business, we would have a problem if we wanted to find out who an employee is, knowing only his salary: we might find a number of entries corresponding to the same salary - we have no function SALARY(EMPLOYEE).

On the other hand, social security numbers are unique to individuals. Hence, in the same business, we could find out from the database the SSN of an employee, or identify an employee from his or her SSN. In other words, we can define two functions: SSN(EMPLOYEE), and EMPLOYEE(SSN). Note that these two functions correspond to reading a table in one direction, or in the opposite direction - but it is the same table. Hence we think of these two functions as *inverse of each other.*

Here is a more mathematical example: take the function $y = x^3$. If you graph this function, you will notice that there is exactly one number that is the cube of another, but that the reverse is also true: any number is the cube of exactly one number. Thus, we can "go in the opposite direction" here too: given y, we can find out the x, such that $x^3 = y$. This number is called the "cube root" of y, and we write $x = \sqrt[3]{y}$. The cube function, and the cube root function are inverse of each other, or, if you prefer, the cube root function is the inverse of the cube function (and vice-versa).

Now we get to a delicate point: take the function $y = x^2$. This function dos not have an inverse: in fact, given y, there are no x's such that $x^2 = y$, if $y < 0$; there is exactly one x, such that $x^2 = y$, if $y = 0$ (that's $x = 0$); but there are two x's such that $x^2 = y$, if $y > 0$. For example, $2^2 = (-2)^2 = 4$, $2.5^2 = (-2.5)^2 = 6.25$, and so on. Now, the problem here is that x^2 is defined for all x, and a number and its opposite have the same square. So, let's perform this trick: we define a "new" function as $f(x) = x^2$, with domain (arbitrarily chosen) $[0, \infty)$. Now, we are only looking at non negative numbers, and, sure enough, given a $y \geq 0$ there is only one non negative number x, such that $x^2 = y$! We usually denote this number by $\sqrt{y}$. Take notice: $\sqrt{y} \geq 0$ always! The other number whose square is y will be the opposite of $\sqrt{y}$, i.e., $-\sqrt{y}$. Thus, note: any positive number x has two square roots: $\sqrt{x}$, and $-\sqrt{x}$.

This idea of "reading" a function in two directions (when possible) should help us realize that the notion of "independent" and "dependent" variable, can be dependent on the context. In our business example, we might be interested in finding SSN of employees, or, in reverse, locating an employee from his/her SSN. And, in any relation, we may have a function "in one direction", "in the other direction", "in both directions", or "in no direction".

Examples:

- The relation $x^2 + y^2 = 1$ is very popular (its graph is a circle of radius 1), and defines no function in either direction.
- The relation $x^2 + y = 1$ defines a function from x to y, but not in the opposite direction.
- The relation $x + y^2 = 1$ defines a function from y to x, but not in the opposite direction.
- The relation $x + y = 1$ defines a function from x to y, as well as an inverse function from y to x (this is true of all *linear* relations)

4 Functions: Combining functions

Again, this is a topic we will go back to in a dedicated file. But we can start working on it right now. A function is a rule. But it may be convenient, sometimes, to think of a rule obtained by combining two (or more) rules in some way. For example, we may consider a function like $4x^2 + 2x$ as obtained by *summing* the function $4x^2$, and the function $2x$. This can be done with any operation -- after all, writing, say, $h(x) = \frac{f(x)}{g(x)}$ means simply: "take x, compute $y = f(x)$, and $z = g(x)$ - $h(x)$ is the ratio $\frac{y}{z}$"

An important case is when we combine two function "in a cascade". For example, the function $\sqrt{2 + x^2}$ can be seen as obtained by, first, computing the function $f(x) = 2 + x^2$, and then applying the function $g(x) = \sqrt{x}$ to the result: $g(f(x))$.

When a function is looked upon as a combination of two functions, its domain is the set of numbers for which both operations make sense. That is, looking at $f(g(x))$, x must be such that $g(x)$ makes sense, *and* $f(g(x))$ does, that is the output of g must be in the domain of f. Thus, in our last example, f makes always sense. Also, its output is necessarily positive. Since the domain of g is $[0, \infty)$, $f(x)$ will always be in the domain of g, and hence the domain of $g(f(x))$ is, in this case, all the real numbers. However, if we looked at, say, $\sqrt{2 - \frac{1}{x}}$, and looked at it as the composition of $f(x) = 2 - \frac{1}{x}$ and $g(x)$, f is only defined for $x \neq 0$ (so 0 has to be excluded from the domain), but g is defined only for non negative numbers, so we also need $f(x) = 2 - \frac{1}{x} \geq 0$, or $\frac{1}{x} \leq 2$, or $x \geq 2$. The last constraint makes the first ($x \neq 0$) redundant, so the domain of the composite function is $[2, \infty)$.