Factoring and Roots

The Relation Between Factoring and Roots of a Polynomial

Let's start with an example. Consider the following function

$$f(x) = 2(x-1)(x+2)(x-4)$$

:

If we expand the products, we see that it is a polynomial, of degree 3. Now, given the zero product rule, we know that the equation

$$f(x) = 0$$

has exactly three solutions: $x = 1, -2, 4$. Right away, we also know that the equation

$$2x^3 - 6x^2 - 12x + 16 = 0$$

(that's what you get if you expand the parentheses) has the same numbers as roots, since it is, after all, the same equation. In general, if a polynomial $p(x)$ has the property that a certain number $\overline{x}$ is such that $p\left(\overline{x}\right) = 0$, $\overline{x}$ is called a **root** of the polynomial. As we see in this example, each root corresponds to a factor in a factoring of the polynomial.

Of course, sometimes, a polynomial of degree n has less than n roots. This may happen for two reasons:

1. Some roots are "missing": for example

   $$x^3 - x^2 + x - 1$$

   has only $x = 1$ as a (real) root. This can be seen (we'll see later how to find this out) because

   $$x^3 - x^2 + x - 1 = (x-1)\left(x^2+1\right)$$

   and the right hand side is equal to zero only when $x = 1$, since $x^2 + 1 > 0$, for any value of x, i.e., it has no roots of its own.

2. Some roots are "double" (or even more): for example

$$x^2 + 2x + 1 = (x+1)(x+1) = (x+1)^2$$

and is zero only if $x = -1$—but, in an obvious sense, this root "counts for two". If a root corresponds to a double, triple,... factor, it is called a double, triple,... root—in general, a *multiple root*.

It turns out that every polynomial can be factored in terms of its roots. This is a very deep and basic theorem, at the foundation of higher algebra (hence, its name: *The Fundamental Theorem of Algebra*):

Theorem (*The same result can be expressed in many ways—here we emphasize the aspect relating to factoring in terms of real numbers*) Every polynomial can be factored in terms of a constant times linear (first degree binomials) and quadratic (second degree trinomials) factors. Specifically, the constant is the coefficient of the highest order term. The monomials are all of the form $(x - a)$, where a is a real root of the polynomial—there is one such factor for every root (a given number may appear more than once, in the case of multiple roots, as in case **??** above). What is left can be factored in second degree polynomials, none of which has real roots. If the polynomial has degree n, and there are k roots (multiple roots are counted according to their multiplicity), necessarily, $k \leq n$, $n - k$ is even, and there are $\frac{n-k}{2}$ quadratics in its factoring.

The point of this theorem is that each real root of a polynomial corresponds to a linear factor in its factoring.