Exponential and Logarithmic Functions

1 Exponential Functions

Take a positive number a (*the base*) and consider its powers, when raised to a positive integer $n = 1, 2, \ldots$ (*the exponent*). That's simply a shorthand for repeated multiplications by a. Looking at the rules for multiplying and dividing powers of the same base, it turns out that it is convenient to extend the definition of powers as follows:

- $a^0 = 1$ for all positive bases a
- $a^{-n} = \frac{1}{a^n}$ for all positive bases a and positive integers n
- $a^{\frac{1}{n}}$ is the *n-th root of* a, meaning the unique positive number b such that $b^n = a$
- $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

Irrational numbers are, intuitively, numbers that can be approximated to any desired degrees by rational numbers, without being rational themselves. The poster children are $\sqrt{2}$ (or any non-exact root) – easy to show as irrational – and π (the ratio of the circumference of a circle to its diameter) – hard to show as irrational. Relying on this approximation feature, we can define a^x for any real number x:

> To define a^x for irrational x we use the fact that irrational numbers can be approximated as well as we wish by rational numbers, so that if x is irrational, it will be approximated as well as we wish by a^r where r is a rational number that approximates x well enough.

A number of nice properties follow, but only because we only considered positive bases. For example,

- $a^x \cdot a^y = a^{x+y}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $(a^x)^y = a^{xy}$

These hold for any $a > 0$ and any real numbers x and y.

Note The rules above fail if they are applied to powers with negative bases. For example, $\left((-1)^2\right)^{1/2} = 1^{1/2} = 1$, but $\left((-1)^{1/2}\right)^2$ is undefined.

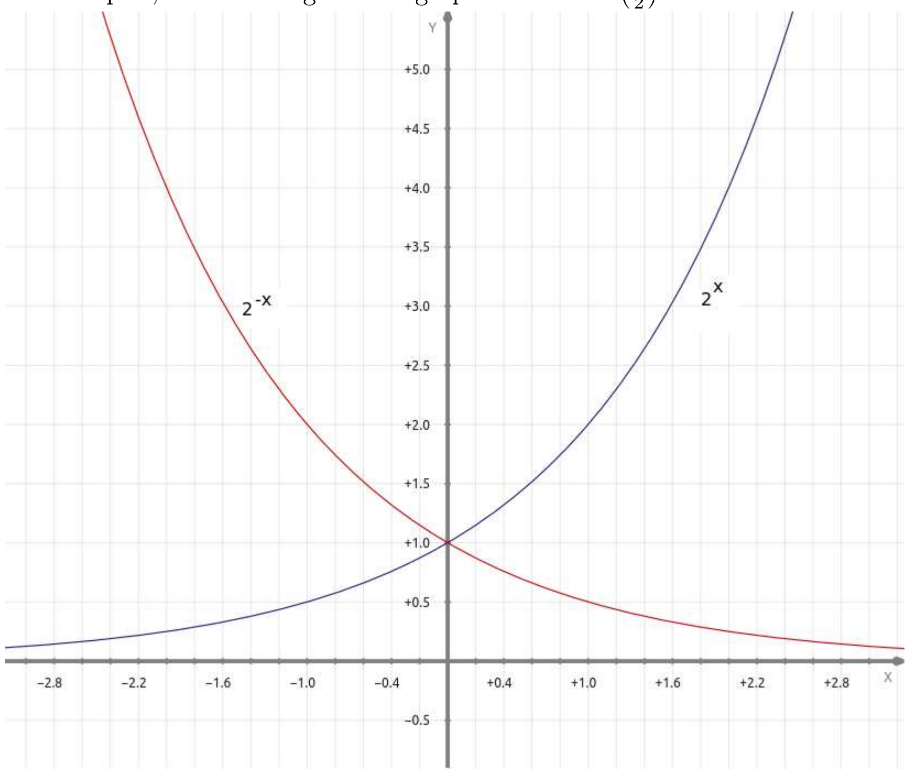
2 Graphs

If $a > 1$, the graph of a^x turns out to be always increasing, and always positive. As x becomes very negative a^x gets closer and closer to 0.

If $a < 1$, then $b = \frac{1}{a} > 1$, and, by the definition of negative exponents, $a^x = b^{-x}$, so the graph is the mirror image of b^x, reflected around the vertical axis.

Checking with a graphing utility, you can see that exponential functions have nice smooth graphs. Hence, if you simply evaluate a^x for simple values of x – basically integers – you can complete the graph by simply joining the points with a nice continuous line.

As examples, the following are the graphs of 2^x and $\left(\frac{1}{2}\right)^x = 2^{-x}$:



3 Logarithmic Functions

Looking at the graph of an exponential function we will notice that no two different values of the input will produce the same output. Functions with this property are called *one-to-one* (one input to one output), and are *invertible*, that is, we can find one input only that will produce a given output. This "reversed" function (we choose the output and find the corresponding input) is called the *inverse function* of our original function. If the original function was called f, the inverse is denoted by f^{-1}

Note This is *not* the same as $\frac{1}{f}$ (the reciprocal function)! It is maybe unfortunate that this notation is standard even if possibly causing some confusion, since the reciprocal of a number a is denoted by $\frac{1}{a}$ but often also by a^{-1}).

Since exponential functions of the form $f(x) = b^x$ are one-to-one, we can define their inverse functions, called *logarithms*[1] $f^{-1}(x) = \log_b(x)$. By definition, if $y = \log_b(x)$, then $b^y = x$, and if $b^x = y$, then $x = \log_b(x)$. You will often see the shorthand notation $\log_b x$ for $\log_b(x)$. b is called the *base* of the logarithmic function $\log_b(x)$. Note that, as $\log_b(x)$ is the inverse function of b^x, b^x is the inverse function of $\log_b(x)$. This is a general fact: $\left(f^{-1}\right)^{-1} = f$!

Since b^x is only defined for $b > 0$, we can only consider positive number as bases. Also, since $b^x > 0$ for any x, we can only consider logarithms of positive numbers. On the other hand, logarithms can take any real value (check their graph if you don't find this statement obvious).

Properties of logarithmic functions follow from the corresponding properties of their inverse functions, exponential functions:

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
- $\log_b\left(x^k\right) = k \cdot \log_b(x)$
- $\log_b(1) = 0$
- $\log_b(b) = 1$
- $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$

for any positive number b and any positive numbers x and y.

Remark In practice, only two bases are used (that's why you only have two log keys in your calculator). The choice $b = 10$ is common in science, where logarithms of constants or measured quantities make it is easy to transition between logarithms and scientific notation. When considering exponential functions (e.g. of time), it turns out that the natural base is an irrational number denoted by e. You will learn about this base, and its advantages, if you take a Precalculus and a Calculus class.

Note In computer science, and information theory, it also common to use 2 as a base for powers and logarithms.

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[1] The exotic name comes from the Greek language, as John Napier, who introduced the notion, combined the Greek words for "ratio" and "number".