Exponentials and Logarithms

1 Exponentials and Logarithms

We will not really provide a precise definition of functions like 2^x. Essentially, if x is rational, we already have a definition. If x is irrational, we can approximate it, as well as we wish, with a rational number (that's what we do when we write a decimal expansion, and stop at some point: the truncated expansion is a rational number, and the farther we go before stopping, the better the approximation). 2^x is the number that is approximated better ad better as we approximate the irrational number x.

One important point to mark is that we define functions like a^x *only for positive* a! It is not hard to figure out why (any time we change x by a little, "hitting" an even root, the expression loses meaning), but it is a crucial limitation. Another important point, which we can check, for now, both intuitively, and by looking at a graph created by a calculator or computer, is that

- $f(x) = a^x$ is always increasing if $a > 1$
- $f(x) = a^x$is always decreasing if $a < 1$

This implies that a^x, for any $a > 0, a \neq 1$, has an inverse. This inverse function is called the *logarithm in base* a.

Hence

$$y = a^x \Leftrightarrow x = \log_a y$$

It turns out that the usual rules for powers apply, no matter what x may be. hence, we have

- $a^x a^y = a^{x+y}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $(a^x)^y = a^{xy}$
- etc.

It is not hard to see that these rules imply a number of rules for logarithms. For example,

- $\log_a (xy) = \log_a x + \log_a y$
- $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a (x^y) = y \log_a x$
- etc.

Exponential and logarithmic functions are immensely useful, and can be found in practically every area of science, finance, and engineering. They are also essential tools in the development of calculus.

2 The most important bases

You might have wondered why your calculator does not have keys for "$\log_a x$", for generic a, but only for $a = 10$, and $a = e$ (Euler's number - more about this later). Fact is these are the only bases used in practice (actually, base 2 is also popular, especially in computer science), and, more importantly, if we have a way of computing logarithms ("logs" in colloquial mathspeak) in a base (any base), we can compute them in any other. Thus we could live with a single log key, and deduce all others logs, as needed (this is precisely what happens within MATLAB, Octave, and similar software packages for high precision numerical computations - if you key in "log(x)", it means $\log_e x$, or $\ln x$, using the notation of the book).

You can do worse than learning the "change of base formula" from the book:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

(by the way, since $\log_a a = 1$, for any a, this formula also implies that $\log_a b = \frac{1}{\log_b a}$). Thus, if you know how to compute, say $\ln 4.765$ (using your calculator, of course), you can easily compute, for example

$$\log_{23} 4.765 = \frac{\ln 4.765}{\ln 23}$$

This does not quite explain the central role of 10, and e in the word of logs. The reasons are very different. Base 10 is extremely convenient if you need to compute logarithms by looking them up in a table, like people had to do before the ubiquitous advent of calculator and computers (i.e., well into the 1970s). This also led to physical and chemical measures, that are conveniently represented by logarithms of basic quantities, to be expressed as base-10 logarithms. That's why to calculate pH in chemistry, and decibels in acoustic, we use base 10 logarithms of more fundamental quantities. Incidentally, the Richter scale for earthquakes is also logarithmic - in base 10 of a quantity related to the energy of the quake (it turns out to be "in base 32" in terms of the energy released by the quake). Since we don't use tables any more, base 10 is not as useful as it used to be, and survives mainly in these applications, referring to constants.

Once we consider *functions,* exponential or logarithmic, it turns out that e is the only base you would want to use. In fact, any math book beyond basic calculus will ignore logarithms in any base but e. This pre-eminence becomes apparent in calculus, where the functions e^x, and $\ln x$ play a huge role, and where it is shown why they get the moniker *natural.* A superficial sketch of how one would come up with e in the first place, in a "natural" way, is presented in another note.