Compound Interest

1 Review Of Compound Interest

Ignoring the "marvelous" workings of compounding interest cost dearly to the landed gentry of Virginia in the 18th Century, when they saw their debts to British and New York creditors balloon beyond their understanding. Some contemporary credit card holders may understand the feeling. The principle is much older, and its development in the Renaissance contributed to many financial problems for debtors ever since, and also to fascinating developments in mathematics.

Let's recap the facts. Suppose you have an investment of C dollars, and you are earning an interest rate of r. For example, \$1000 at 5%. With simple interest, your money grows linearly: after t years (where t is any real number - fractions of a year are perfectly meaningful), you have

$$C + Crt = C\left(1 + rt\right) \tag{1}$$

This is a linear function with slope Cr.

Now, compound interest means that, after a set amount of time (usually no more than year, possibly much less), all interest earned up to that point is absorbed in the "principal": from this time on, interest will be computed not on the original investment C, but on C+all interest earned to that point. Suppose this happens after 1/n-th of a year ($n = 1$, after a year;$n = 2$, after 6 months, $n = 12$, after one month, and so on). The same scenario is repeated after the next 1/n-th of a year, and on and on.

This rule produces a much faster growth of your money. Let's see how so, working the formulas.

After the first compounding period, $t = \frac{1}{n}$, and we have, from (1), a total amount of

$$C_1 = C\left(1 + \frac{r}{n}\right)$$

For the next 1/n-th of the year, we will earn interest on this amount. Hence for t between $\frac{1}{n}$, and $\frac{2}{n}$, our money will be

$$C_1\left(1 + rt\right) = C\left(1 + \frac{r}{n}\right)\left(1 + r\left[t - \frac{1}{n}\right]\right)$$

(interst is earned starting at $t = \frac{1}{n}$, hence the strange $t - \frac{1}{n}$ term). At the next compounding time, $\frac{2}{n}$, i.e. $\frac{1}{n}$ from the first, we will have

$$C_2\left(1 + \frac{r}{n}\right) = C\left(1 + \frac{r}{n}\right)\left(1 + \frac{r}{n}\right) = C\left(1 + \frac{r}{n}\right)^2$$

You may see an emerging pattern, because now we have to repeat the same argument again, starting from $t = \frac{2}{n}$. After k compounding periods, we will end up with a total wealth of

$$C\left(1+\frac{r}{n}\right)^k \tag{2}$$

Since each compounding period is $\frac{1}{n}$ long, k compounding periods correspond to a total time $t = \frac{k}{n}$, so that, for values of t that are multiples of $\frac{1}{n}$, formula (2) means that your wealth will be

$$C\left(1+\frac{r}{n}\right)^{nt} \tag{3}$$

Notice that this formula is exact *only for times t that are integer multiples of the compounding period!* Between such times, your wealth will be growing linearly, starting from your accumulated wealth at the last compounding period[1].

It is clear that, the higher the value of n (the shorter the compounding period), the faster your wealth will grow. You may, if you wish, play a bit with your calculator, and get a feeling for the effect that changing the value of n has.

2 Continuous Compounding

The following question arose when banking arose as a legitimate (and very profitable) activity in the early Renaissance: since, clearly, as n increases, your yield will grow, how far can you push this? Maybe, by increasing n, we could push increase our profit indefinitely? This is not a vey simple question to answer, but it turns out that the answer is *no*.

This is a fact that you might want to "check" on a calculator, but its proof requires some serious beginning calculus:

> As n increases, the expression
>
> $$\left(1+\frac{r}{n}\right)^n$$
>
> increase, but always stays (well) below 3. As a metter of fact, there is an irrational number, which we will call e (in honor of the Swiss 18th Century mathematician Leonard Euler, one of the most brilliant mathematicians of all time), such that $\left(1+\frac{r}{n}\right)^n$ will be as close to e^r as we wish, provided we choose a large enough value for n.

Using this fact, we note that (3) can be written as

$$C\left[\left(1+\frac{r}{n}\right)^n\right]^t$$

[1] Algebra textbooks tend to forget to stress this point. On the other hand, since we are not in an Accounting, course, this is not a terrible mistake. However, if we keep this fine detail in mind, we can learn something very interesting about the function e^{rt} - see the discussion in the companion file on the number e.

and it is only reasonable (again, there's some serious math to employ to make the argument air-tight) to decide that this, for large enough n, should be almost equal to

$$Ce^{rt} \tag{4}$$

Such a compounding schedule is called "continuous compounding".

Once somebody is nice enough to program our calculator or computer with the function e^x, formula (4) is much more convenient than (3) for calculations. But, wait, there's more: this formula holds for any t, *not only* for special values, like the multiples of some fixed number. Hence, we have a legitimate exponential funciton to work with, not a complicated function, linear over short intevals, with exponential increases of the slope at each compounding period.

While (4) is not really used by your bank, it is so much more convenient that it is the standard model for interest-driven growth in all financial studies.