Average Rate of Change for Quadratic Functions

Quadratic Functions Revisited

1 Linear Functions

We noticed how linear functions are the precisely those functions whose average rate of change is always the same, no matter where we compute it. This is a very useful feature of linear functions, but it may spark the question: if the average rate of change of a non linear function is not constant, how does it behave? As it happens all so often, this question is easily answered if we look at polynomials.

2 Quadratic Functions

Consider a quadratic function

$$f(x) = ax^2 + bx + c \tag{1}$$

Its average rate of change between two values of x, say x_1 and x_2, depends on both x_1 and x_2. We may equivalently say that it depends on x_1 and $x_2 - x_1$. To get a handle on how this average rate of change behaves, let us vary x_1, and keep the difference with the second point constant - we'll call it Δx (thus, $x_2 = x_1 + \Delta x$). Then, the average rate of change between x and Δx is

$$\begin{aligned} \frac{a(x+\Delta x)^2 + b(x+\Delta x) + c - ax^2 - bx - c}{\Delta x} &= \frac{a\left[(x+\Delta x)^2 - x^2\right] + b\Delta x}{\Delta x} = \\ &= \frac{a\left(2x\Delta x + [\Delta x]^2\right) + b\Delta x}{\Delta x} = \\ &= 2ax + b + a\Delta x \end{aligned} \tag{2}$$

Hence, for fixed Δx, the average rate of change is *linear* as varies, and *its* slope is $2a$. In other words, the coefficient a gives us an idea of the "speed" at which the average rate of change changes as we move along the graph.

3 Beyond Quadratic Functions

Now, what if we look at, say, cubic functions? Suppose

$$f\left(x\right)=ax^3+bx^2+cx+d$$

Then, similarly to the quadratic case,

$$\frac{a\left[\left(x+\Delta x\right)^3-x^3\right]+b\left[\left(x+\Delta x\right)^2-x^2\right]+c\Delta x}{\Delta x}=$$

$$=\frac{a\left(3x^2\Delta x+3x\left[\Delta x\right]^2+\left[\Delta x\right]^3\right)+b\left(2x\Delta x+\left[\Delta x\right]^2+c\Delta x\right)}{\Delta x}=$$

$$=3ax^2+2bx+c+\left(3ax+b\right)\Delta x+\left[\Delta x\right]^2$$

Now, you can see the pattern: if you compute the average rate of change (for fixed Δx) of a polynomial of degree n, as we move along the graph, it will change as a function of x that is a polynomial of degree $n-1$.

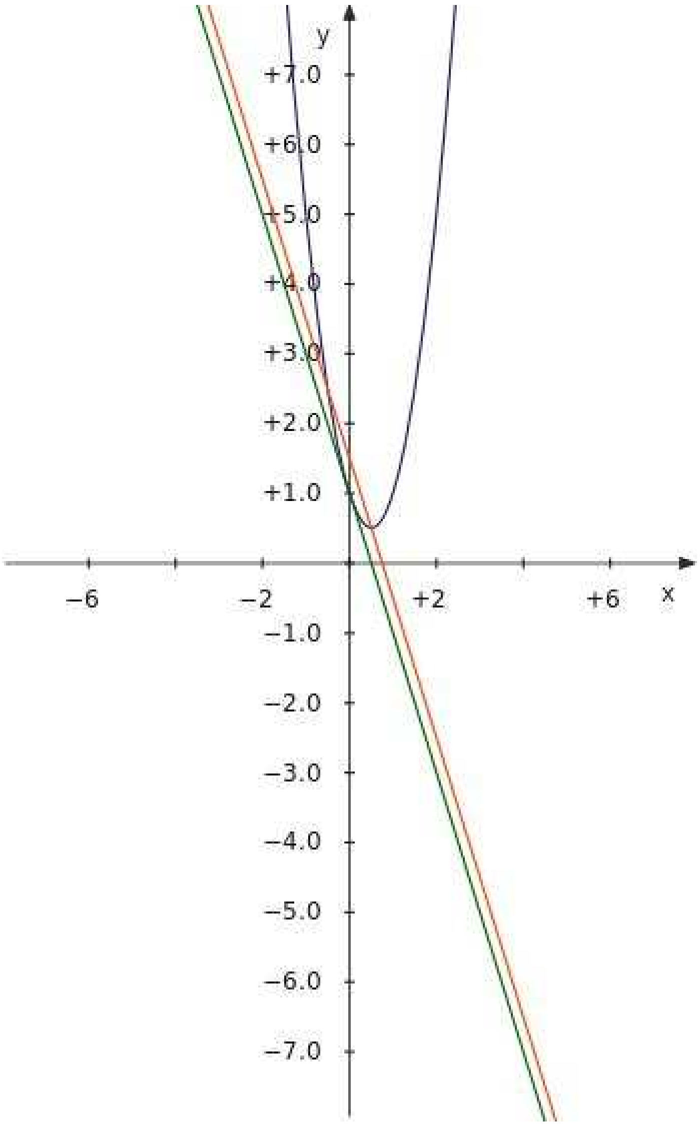
4 There's Even More Amazing Stuff...

What is really surprising (or, in retrospect, maybe not - but it takes a lot more work and a lot more math to dispel the surprise) is that the above discussion connects very closely with the discussion on how to find the tangent line to the graph of a polynomial at any point!

More precisely, looking at the formulas above, for example (2), we may note that, if we choose our two points very close to each other, Δx will be small, and the main contribution to the average rate of change at x is given by $2ax+b$. Similarly, in the cubic case, for Δx small, most of the average rate of change is given by $3ax^2+2bx+c$. Now, if we try to find the slope of the tangent to the quadratic function (1) when $x=k$, we need to shift the graph so that $x=k$ shifts to $x=0$. If $k<0$ we have to shift to the right, that is by a negative value - since $k<0$, by k. If $k>0$ we have to shift to the left, that is by a positive value - since $k>0$, again by k. In both case, we get

$$a\left(x+k\right)^2+b\left(x+k\right)+c=ax^2+\left(2ak+b\right)x+ak^2+bk+c$$

The linear term tells us, as we know, the slope of the tangent at $x=0$ for the shifted curve, i.e. the slope at $x=k$ for the original curve: $2ak+b$ - precisely the average rate of change of the function, up to a term in Δx, which is small if we compute the average over a small interval. No surprise: we just observed that the secant connecting two nearby points is very close to the tangent, something we readily believe when we look at a picture.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