Quick Summary of Asymptotes

Vertical Asymptotes:

If you have a function involving fractions, you have always to check if there are values of the independent variable that would cause a denominator to be zero. Such values have to be excluded, because there is no way to define what "dividing by zero" should mean and keep the standard rules of arithmetic working consistently. This is one of the items to check when looking for the "domain" of a function The domain of a function are those values of the independent variable such that the function can be calculated there - if for a certain value, a denominator is zero, this value is excluded from the domain.

For example, take

$$f(x) = \frac{4x^2 - x + 1}{5x - 1}$$

If $x = \frac{1}{5}$, the denominator is zero. Hence, the domain is given by all numbers, except $\frac{1}{5}$. This can be written in formulas as

$$\left\{ x \middle| x \neq \frac{1}{5} \right\}$$

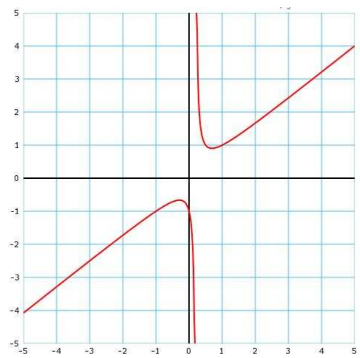
or

$$\left(-\infty, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \infty\right)$$

among others.

Now, we might wonder what happens when x is not equal to $\frac{1}{5}$, but very close to it. In this case, the numerator will be close to $4 \cdot \left(\frac{1}{5}\right)^2 - \frac{1}{5} + 1 = 1 - \frac{1}{25} = \frac{24}{25}$, and the denominator will be very close to zero. Now, a number, like $\frac{24}{25}$ or close to it, divided by a very small number, will be either positive or negative, but with a very large absolute value. Think, for example, $\frac{1}{10^{10}} = 10^{-10} = 0.0000000001$, to get the idea. Hence, the graph of this function will jump very far up or down, as it gets closer and closer to the vertical line $x = \frac{1}{5}$.

This is what's called a "vertical asymptote". In our case, the graph would like this:



The two parts of the curve that fall and climb sharply near $x = \frac{1}{5}$ are neatly separated by a vertical line, whose equation is $x = \frac{1}{5}$.

In general, you would look for all such situations: the denominator is getting close to zero, and the fraction becomes huge - the vertical line over the zero of the denominator (where the function is not even defined) is approached by the graph in the same way as in the picture above.

Horizontal and Oblique Asymptotes

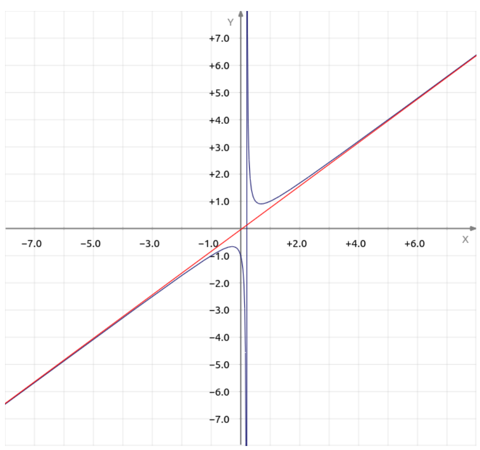
These are different beasts. If you have a rational function, the quotient of two polynomials, and you look at its graph for large values of $|x|$ (far in the positive or far in the negative direction), the only part of the polynomials that will count is the highest power, because the other terms will be much smaller. Hence, very far to the right or left, the function will look very much like the quotient of the two highest powers. This quotient may be a horizontal line, or an oblique line, or no line at all. In the first two cases, one speaks of, respectively, a horizontal or an oblique asymptote.

In our previous example, the ratio in question would be

$$\frac{4x^2}{5x} = \frac{4}{5}x$$

Hence, far to the right or left, the graph of f will be similar to the graph of $g(x) = \frac{4}{5}x$ (as you can see from the picture). To be precise, looking at the highest powers shows us only the slope of the oblique asymptote. To get the equation of the line, we have to compute the quotient of the two polynomials, as discussed in the files on division. In this case, the result is

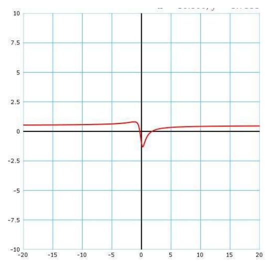
$$\frac{4x^2 - x + 1}{5x - 1} = \frac{4}{5}x - \frac{1}{25} + \frac{24/25}{5x + 1}$$



If the two polynomials had been of the same degree, the ratio would be close to a constant, hence would lead to a horizontal asymptote. For example

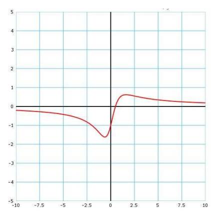
$$h(x) = \frac{2x^2 - 3x - 1}{4x^2 + 1} \approx \frac{2x^2}{4x^2} = \frac{1}{2}$$

and the function $h(x) = \frac{1}{2}$ (a constant) is a horizontal asymptote:



A horizontal asymptote appears also when the numerator is of lesser degree than the denominator. In this case the ratio does not look like a line in the long run, but it is a function that, far to the left or to the right, is closer and closer to zero - hence the asymptote is the $x-$axis, the line $y = 0$. For example, look at

$$q(x) = \frac{2x-1}{x^2+1} \approx \frac{2x}{x^2} = \frac{2}{x}$$



No Asymptote

If the degree of the numerator is at least 2 more than that of the denominator, then the ratio will look like a power higher than 1, that is it will not look like a line, as we say that there's no asymptote. For example,

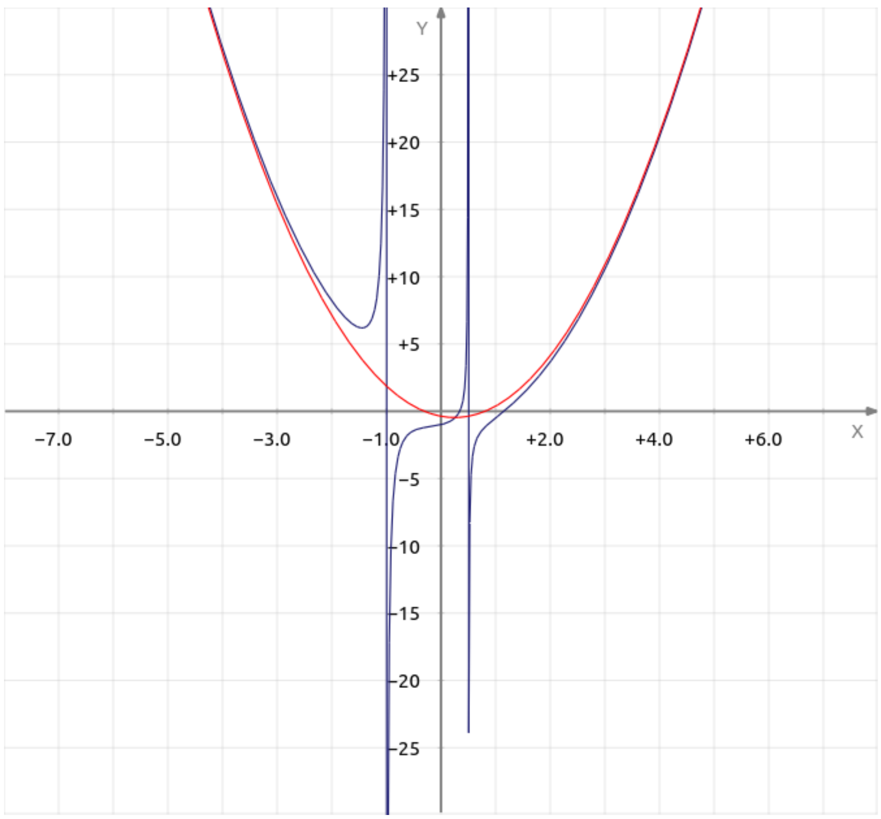
$$\frac{3x^4 - 3x^2 - 2x + 1}{2x^2 + x - 1} \approx \frac{3x^4}{2x^2} = \frac{3}{2}x^2$$

(this has vertical asymptotes at $x = -1$ and $x = \frac{1}{2}$) so the long term behavior will be similar to that of a parabola with leading coefficient $\frac{3}{2}$.

Note Here too, if we wanted to have a sharp approximation for the behavior of the function as $|x| \to \infty$, we should compute the quotient of the two polynomials, beyond the first term. In our example, this is

$$\frac{3x^4 - 3x^2 - 2x + 1}{2x^2 + x - 1} = \frac{3}{2}x^2 - \frac{3}{4}x - \frac{3}{8} + \frac{-\frac{19}{8}x + \frac{5}{8}}{2x^2 + x - 1}$$

so that the more precise long range behavior is that of the parabola $\frac{3}{8}\left(4x^2 - 6x - 3\right)$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