Asymptotes

Rational functions may display a new behavior. A linear function, as we know, is defined for all values of the variable, and increase without bound on one end, decreasing without bound on the other, unless it is a constant. We will soon see that quadratic functions increase or decrease without bound on both ends. In fact, it can be shown that all polynomials are defined for any value of the variable, and, as x becomes larger and larger or smaller (in the sense of more and more negative), they grow or decrease without bound.

Rational functions may behave similarly, but they often do not. There are two issues: they may not be defined everywhere, and, as x grows, or decreases without bound they may well tend to stabilize close to a value, rather than grow or decrease without bound. Near values where the function is undefined, its absolute value grows without bound (making it very tricky to evaluate with a calculator: calculator are not very good at handling numbers whose absolute value is very large or very small).

All of this is sometimes tricky to discover using a graphing calculator - all sorts of fake artifacts may appear (the book hints at the most egregious one on page 407). This is another example where knowing how to handle things algebraically (and with common sense) beats blind reliance on a machine. Let us discuss some examples.

1 A "Tame" Rational Function

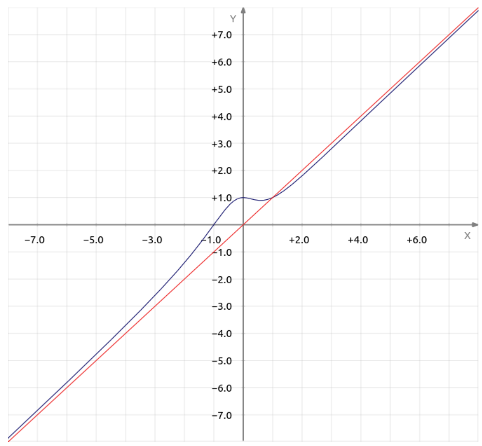
Consider the function

$$f(x) = \frac{x^3+1}{x^2+1}$$

(it is irreducible: thanks to a well known special product $x^3+1 = (x+1)\left(x^2-x+1\right)$, so that there is no common factor). Now, the function is always defined, since $x^2+1 \geq 1$, and, as x grows in absolute value, 1 becomes negligible compared to both x^2 and x^3. Hence, the function will eventually behave roughly like $\frac{x^3}{x^2} = x$, that is will behave (for large $|x|$) almost like a linear function. To be more precise, we need to perform the division, ending up with

$$\frac{x^3+1}{x^2+1} = x + \frac{1-x}{x^2+1}$$

so that, indeed, in the log run, the function will behave like $y = x$:



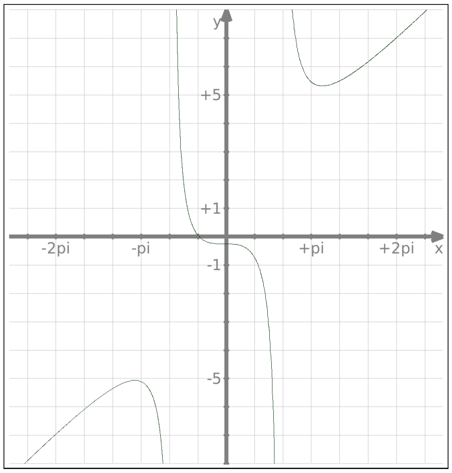
2 A “Hole “ In The Domain

Let’s change the function a little:

$$g(x) = \frac{x^3 + 1}{x^2 - 4}$$

The argument above about what happens when $|x|$ is very large is the same (4 is just as negligible as 1, when confronted with a large $|x|$), but now the function is undefined when $x^2 - 4 = 0$, i.e. $x^2 = 4$, which is true if $x = 2$, or $x = -2$.

Now, when we chose a value very close to either 2 or -2, we have an interesting behavior. Take, say, x very close to 2. Then, $x^3 + 1$ is going to be very close to 9, but $x^2 - 4$ will be very close to 0. Now, a number like 9, divided by a very small number will result in a huge absolute value. As a matter of fact, if x is close to 2, and less than 2, $x^2 - 2 < 0$, but really close to 0. So we have something approximately equal to 9, divided by a negative number with a really small absolute value, resulting in a negative number with a really large absolute value. If $x > 2$, but very close, we will have a similar behavior, with a positive result, still with a very large absolute value. In fact, the absolute value of $g(x)$ grows without bound as we approach 2 from either side. In some sense, the vertical line $x = 2$ acts as a separator between two parts of the graph, with the graph getting closer and closer to this line, without ever touching it (2 is *not* in the domain of g). The line $x = 2$ is called a *vertical asymptote*. The same feature (with small adaptations) is observed if x is very close to -2, and this function has two vertical asymptotes: $x = -2$, and $x = 2$. A graph gives an idea (but cannot conclusively prove that we do have an asymptote - as far as your calculator knows, the graph might climb very high, but then come down):



By the way, a calculation just as in section 1 shows that for large $|x|$ this function will behave like $y = x$.

3 Stabilizing

Consider now a different variation on our original function:

$$h(x) = \frac{x^2 - 1}{x^2 + 1}$$

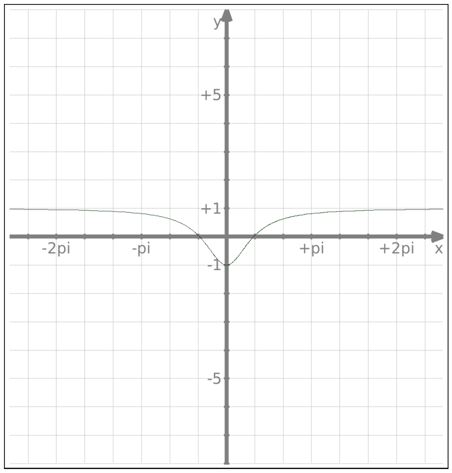
(it's still irreducible: $x^2 + 1$ cannot be factored). Like the case of f in section 1, this function is defined everywhere. However, if we now think of x as having a large absolute value, so that 1 becomes almost negligible, h will look very much like $\frac{x^2}{x^2} = 1$, that is, like a constant. In other words $|h|$ will not grow without bound at all. It will actually tend to lie down horizontally, looking very much like the constant function $y = 1$. The line $y = 1$ will thus be approached closer and closer by h, as $|x|$ grows[1]. Again, a graph helps understand what is going on, but, in principle, for all your calculator knows, for extremely large x the graph might suddenly start to jump all over the place - the range of numbers your calculator understands is large, but finite):

[1] In principle, the graph could cross the line $y = 1$ somewhere, which in this case it doesn't, but in any case, "in the long run" the graph of h will approach but never again touch the line $y = 1$. In fact, suppose that for some value of x, $h(x) = 1$. Then

$$\frac{x^2 - 1}{x^2 + 1} = 1$$

$$x^2 - 1 = x^2 + 1$$

which cannot be ever true, or we would have that $-1 = 1$!

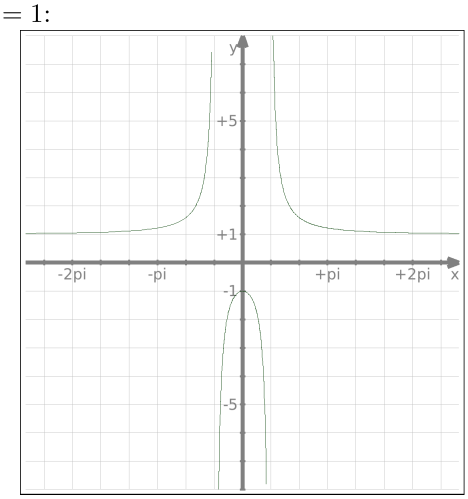


4 All Of The Above

Nothing prevents a rational function to have both types of asymptote. For example,

$$k(x) = \frac{x^2+1}{x^2-1}$$

has two vertical asymptotes, $x = -1$, and $x = 1$, and the horizontal asymptote $y = 1$:



5 Finally, A Rule Of Thumb

How can we spot without too much work what, if any, asymptotes there may be? It's easier than you might think. In fact we have this easy rule of thumb (you can convince yourself that it works by trying to mimic the arguments in the preceding sections):

1. If there are values of x that cause the denominator to be zero, while the numerator is not zero, there is a vertical asymptote, corresponding to that value.

2. If the numerator has a higher degree than the denominator, there are no horizontal asymptotes. If the degree of the numerator exceeds the degree of the denominator by 1, there is an oblique asymptote (to be determined by dividing the two polynomials).

3. If the numerator degree less than or equal to the denominator, there is a horizontal asymptote (if the denominator has higher degree, the asymptote is $y = 0$, if they have the same degree, the asymptote is given by $y = c$, where c is the ratio between the coefficients of the highest powers in the numerator and the denominator[2]

[2] So, for example,

$$\frac{3x^4 + 4x^2 - 2}{5x^4 + 3}$$

has no vertical asymptotes ($5x^4 + 3 \geq 3$), and has a horizontal asymptote, $y = \frac{3}{5}$