Average Rate of Change for Trigonometric Functions

Part I. Sine and Cosine

1 Direct Computation

1.1 Addition Formulas

Using the addition formulas , we find

$$\frac{\sin(x+h)-\sin(x)}{h} = \frac{\sin x\cos h+\sin h\cos x-\sin x}{h} = \sin x\frac{\cos h-1}{h}+\cos x\frac{\sin h}{h} \quad (1)$$

In the same spirit

$$\frac{\cos(x+h)-\cos(x)}{h} = \frac{\cos x\cos h-\sin x\sin h-\cos x}{h} = \cos x\frac{\cos h-1}{h}-\sin x\frac{\sin h}{h} \quad (2)$$

1.2 Sum-Product Formulas

Since most people are more familiar with the addition formulas than the "product-sum" formulas, the previous calculations are the ones you are most likely to find (e.g., in your next Calculus class). However, the product-sum formulas are just as good, if not better: for example, since $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$, we have that

$$\sin(x+h)-\sin(x) = 2\sin\left(\frac{x+h-x}{2}\right)\cos\left(\frac{x+h+x}{2}\right) = 2\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)$$

Hence,

$$\frac{\sin(x+h)-\sin(x)}{h} = \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\cos\left(x+\frac{h}{2}\right) \quad (3)$$

Similarly, since $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

$$\frac{\cos(x+h)-\cos(x)}{h} = -2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right) = -\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\sin\left(x+\frac{h}{2}\right) \quad (4)$$

2 Casting The Formulas In a Better Form

Both formulas (1) and (2) are in the form $A\cos x + B\sin x$. These can always be rewritten in terms on a single trigonometric function at the "price" of introducing a phase. The trick is to find an angle y such that

$$A = C\sin y, \quad B = C\cos y$$

or an angle z such that

$$A = D\cos z, \quad B = D\sin z$$

By the Pythagorean Identity, the coefficients C or D must be such that

$$C^2 = D^2 = A^2 + B^2$$

that is

$$C = D = \sqrt{A^2 + B^2}$$

(we could, equivalently, choose the negative square root, of course). Consequently, we must satisfy

$$\sin y = \frac{A}{\sqrt{A^2 + B^2}}, \cos y = \frac{B}{\sqrt{A^2 + B^2}}$$

or

$$\cos z = \frac{A}{\sqrt{A^2 + B^2}}, \sin z = \frac{B}{\sqrt{A^2 + B^2}}$$

result in, respectively, in

$$C\sin(x + y)$$

and

$$D\cos(x - z)$$

Applying these formulas to our case, we have that

$$\sin x\frac{\cos h - 1}{h} + \cos x\frac{\sin h}{h} = A\sin x + B\cos x$$

with $A = \frac{\cos h - 1}{h}, B = \frac{\sin h}{h}$. Hence, $A^2 = \frac{1 + \cos^2 x - 2\cos x}{h^2}, B = \frac{\sin^2 h}{h^2}$, and $A^2 + B^2 = \frac{1 + \cos^2 h + 2\sin h - 2\cos h}{h^2} = 2\frac{1 - \cos h}{h^2}$. Finally,

$$\frac{A}{\sqrt{A^2 + B^2}} = \frac{\cos h - 1}{2\sqrt{1 - \cos h}} = -\frac{1}{2}\sqrt{1 - \cos h}$$

$$\frac{B}{\sqrt{A^2 + B^2}} = \frac{\sin h}{2\sqrt{1 - \cos h}}$$

Hence, if we define

$$\epsilon = \sin^{-1}\left(-\frac{1}{2}\sqrt{1-\cos h}\right) = \cos^{-1}\left(\frac{\sin h}{2\sqrt{1-\cos h}}\right)$$

we can write

$$\frac{\sin(x+h)-\sin(x)}{h} = \frac{\sqrt{2(1-\cos h)}}{h}\cos(x+\epsilon) \tag{5}$$

could get similar formulas for the cosine, as well as the same in terms of a cosine, instead of a sine.

3 The Case When h is Very Small

The main observation that allows us to get a good feel for the situation when $h \approx 0$ is the geometric observation (look at the unit circle, a very small angle, and a little intuition – the argument will be made precise very early in your first Calculus class) that for h very small,

$$\sin h \approx h$$

Hence,

$$\frac{\sin h}{h} \approx 1$$

Consequently,

$$\frac{\sin^2 h}{h} \approx \frac{h^2}{h} = h \approx 0$$

It follows from this that

$$0 \approx \frac{\sin^2 h}{h} = \frac{1-\cos^2 h}{h} = \frac{(1-\cos h)(1+\cos h)}{h} = (1+\cos h)\frac{1-\cos h}{h} \approx 0$$

Since $1+\cos h$ is, at most, equal to 2, the really small value implies that, at this level of approximation (that is $h \approx 0$)

$$\frac{1-\cos h}{h} \approx 0$$

In particular, this implies that, at this level of approximation, $\cos h \approx 1$, and, as a side consequence,

$$\tan h = \frac{\sin h}{\cos h} \approx \frac{h}{1} = h \tag{6}$$

Inserting these approximation in (1) and (2), results in

$$\frac{\sin(x+h)-\sin(x)}{h} = \sin x\frac{\cos h - 1}{h} + \cos x\frac{\sin h}{h} \approx \cos x \tag{7}$$

$$\frac{\cos(x+h)-\cos(x)}{h} = \cos x\frac{\cos h - 1}{h} - \sin x\frac{\sin h}{h} \approx -\sin x \tag{8}$$

You may notice that we get to the same results much faster if we start from the expressions obtained through the "product-sum" formulas. Looking at (3), we end up with (7), if we observe that $h \approx 0$, and $\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \approx 1$. Similarly, looking at (4), we find (8) by the same arguments.

Remark: We have already argued that $\frac{1-\cos h}{h} \approx 0$. If we compare the approximation in (7) with the expression in (5), we see that it must be $\epsilon \approx 0$, and $\frac{\sqrt{2(1-\cos h)}}{h} \approx 1$, when $h \approx 0$. That implies

$$2\,(1-\cos h) \approx h^2$$

$$1-\cos h \approx \frac{h^2}{2}$$

or

$$\cos h \approx 1 - \frac{h^2}{2}$$

Part II. Tangent and Cotangent

The formulas for the tangent and the cotangent do not look as elegant, as long as we don't go to the approximation for small h :

$$\frac{\tan(x+h)-\tan x}{h} = \frac{1}{h}\left[\frac{\tan x + \tan h}{1-\tan x \tan h} - \tan x\right] =$$

$$= \frac{1}{h}\left[\frac{\tan x + \tan h - \tan x + \tan^2 x \tan h}{1-\tan x \tan h}\right] = \frac{1}{h}\left[\frac{\tan h\left(1+\tan^2 x\right)}{1-\tan x \tan h}\right] =$$

$$= \frac{\tan h}{h} \cdot \frac{1+\tan^2 x}{1-\tan x \tan h} \tag{9}$$

The identities we listed did not include a formula for the cotangent of the sum of two angles, but it's easy to get one

$$\cot(x+y) = \frac{1}{\tan(x+y)} = \frac{1-\tan x \tan y}{\tan x + \tan y} = \frac{1-\frac{1}{\cot x \cot y}}{\frac{1}{\cot x}+\frac{1}{\cot y}} = \frac{\frac{\cot x \cot y - 1}{\cot x \cot y}}{\frac{\cot y + \cot x}{\cot x \cot y}} =$$

$$\frac{\cot x \cot y - 1}{\cot x + \cot y}$$

(note the similarity with the tangent formula). Consequently

$$\frac{\cot(x+h)-\cot x}{h} = \frac{1}{h}\left[\frac{\cot x \cot h - 1}{\cot x + \cot h} - \cot x\right] = \frac{1}{h}\left[\frac{\cot x \cot h - 1 - \cot x\,(\cot x + \cot h)}{\cot x + \cot h}\right] =$$

$$= \frac{1}{h}\left[-\frac{1+\cot^2 x}{\cot x + \cot h}\right] \tag{10}$$

If we now assume h to be very small, we will have, as observed in (6) that $\tan h \approx h$. Looking at (9), we have then that

$$\frac{\tan h}{h} \cdot \frac{1 + \tan^2 h}{1 - \tan x \tan h} \approx 1 \cdot \frac{1 + \tan^2 x}{1 - h \tan x} \approx 1 + \tan^2 x = \frac{1}{\cos^2 x}$$

(the last step is an identity you may find in any list).

Going now to (10), we have

$$\frac{1}{h}\left[-\frac{1 + \cot^2 x}{\cot x + \cot h}\right] \approx -\frac{1 + \cot^2 x}{h \cot x + \frac{h}{\tan h}} \approx -\frac{1 + \cot^2 x}{1} = -\left(1 + \cot^2 x\right) = -\frac{1}{\sin^2 x}$$

The last step follows easily:

$$1 + \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$