Construction Proof for $\sin(\alpha+\beta) = \ldots$ and $\cos(\alpha+\beta) = \ldots$

Prove by construction that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, if α and β are positive acute angles.

Although this proof is for the case when $0° < \angle\alpha + \angle\beta$ and $\angle\alpha$ and $\angle\beta$ are both positive, it can be extended to any $\angle\alpha$ and $\angle\beta$.

Angle α is in the standard position. Angle β is added to angle α. The proof is divided into two cases.

Case 1: $0° < \angle\alpha + \angle\beta \leq 90°$

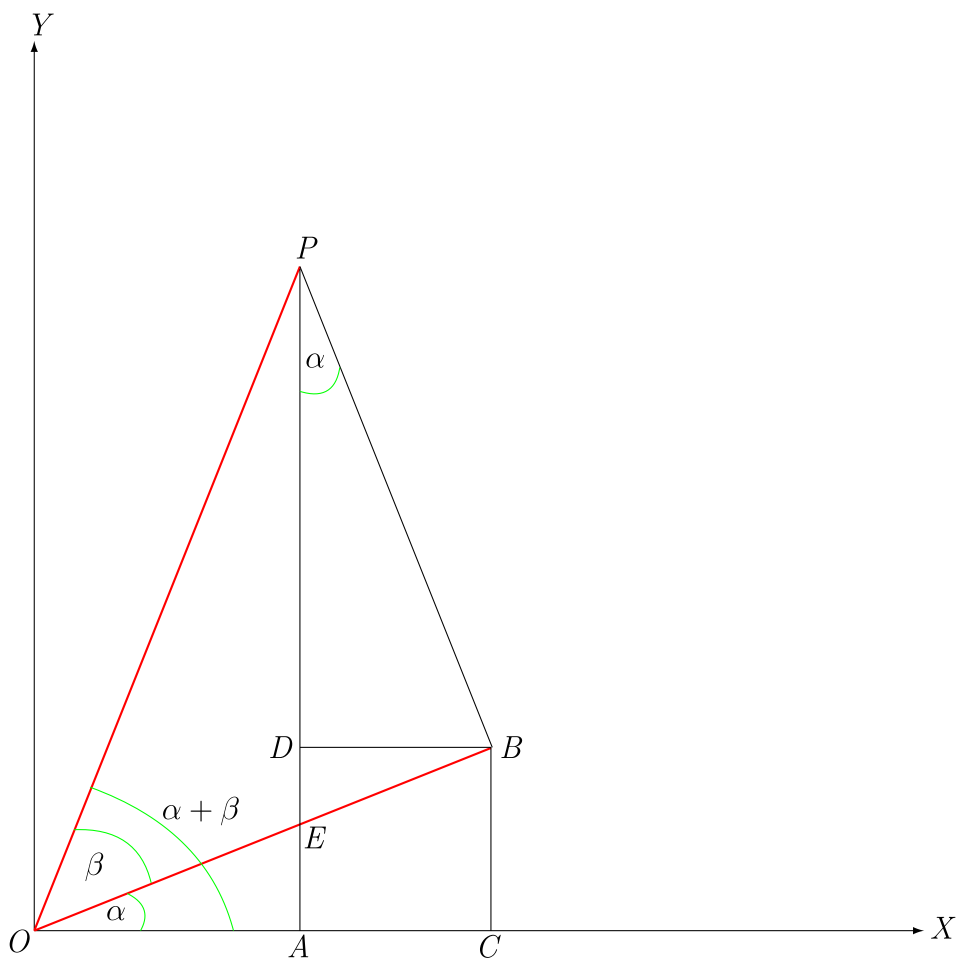

Figure 1: $0° < \angle\alpha + \angle\beta \leq 90°$

Let P be any point on the terminal side of $\angle\alpha + \angle\beta$. Draw PA perpendicular to OX. Draw PB perpendicular to the terminal side of $\angle\alpha$; BC perpendicular to OX; and BD perpendicular to PA. Line OB intersects line PA at point E.

We now have to prove that $\angle\alpha = \angle APB$.

By vertical angles, $\angle AEO = \angle PEB$. By construction, $\angle PAO$ and $\angle PBO$ are both right angles. Since the angles of a triangle sum to 180°, $\angle AOE$ in $\triangle AOE$ is equal to $\angle EPB$ in $\triangle EPB$; therefore $\angle\alpha = \angle APB$.

The rest of the proof consists of basic definitions and algebra:

$$\begin{aligned}\sin(\alpha+\beta) &= \frac{AP}{OP} = \frac{AD+DP}{OP} = \frac{CB+DP}{OP} = \frac{CB}{OP} + \frac{DP}{OP} \\ &= \left[\frac{CB}{OB} \times \frac{OB}{OP}\right] + \left[\frac{DP}{BP} \times \frac{BP}{OP}\right]\end{aligned}$$

By relating the constructed figure to the last expression in the above equality, we can see that:

- $\frac{CB}{OB} = \sin\alpha$
- $\frac{OB}{OP} = \cos\beta$
- $\frac{DP}{BP} = \cos\alpha$
- $\frac{BP}{OP} = \sin\beta$

Therefore $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

For the cosine formula:

$$\begin{aligned}\cos(\alpha+\beta) &= \frac{OA}{OP} = \frac{OC-AC}{OP} = \frac{OC-DB}{OP} = \frac{OC}{OP} - \frac{DB}{OP} \\ &= \left[\frac{OC}{OB} \times \frac{OB}{OP}\right] - \left[\frac{DB}{BP} \times \frac{BP}{OP}\right]\end{aligned}$$

Then:

- $\frac{OC}{OB} = \cos\alpha$
- $\frac{OB}{OP} = \cos\beta$

- $\dfrac{DB}{BP} = \sin\alpha$
- $\dfrac{BP}{OP} = \sin\beta$

and so $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Case 2: $90^\circ < \angle\alpha + \angle\beta < 180^\circ$

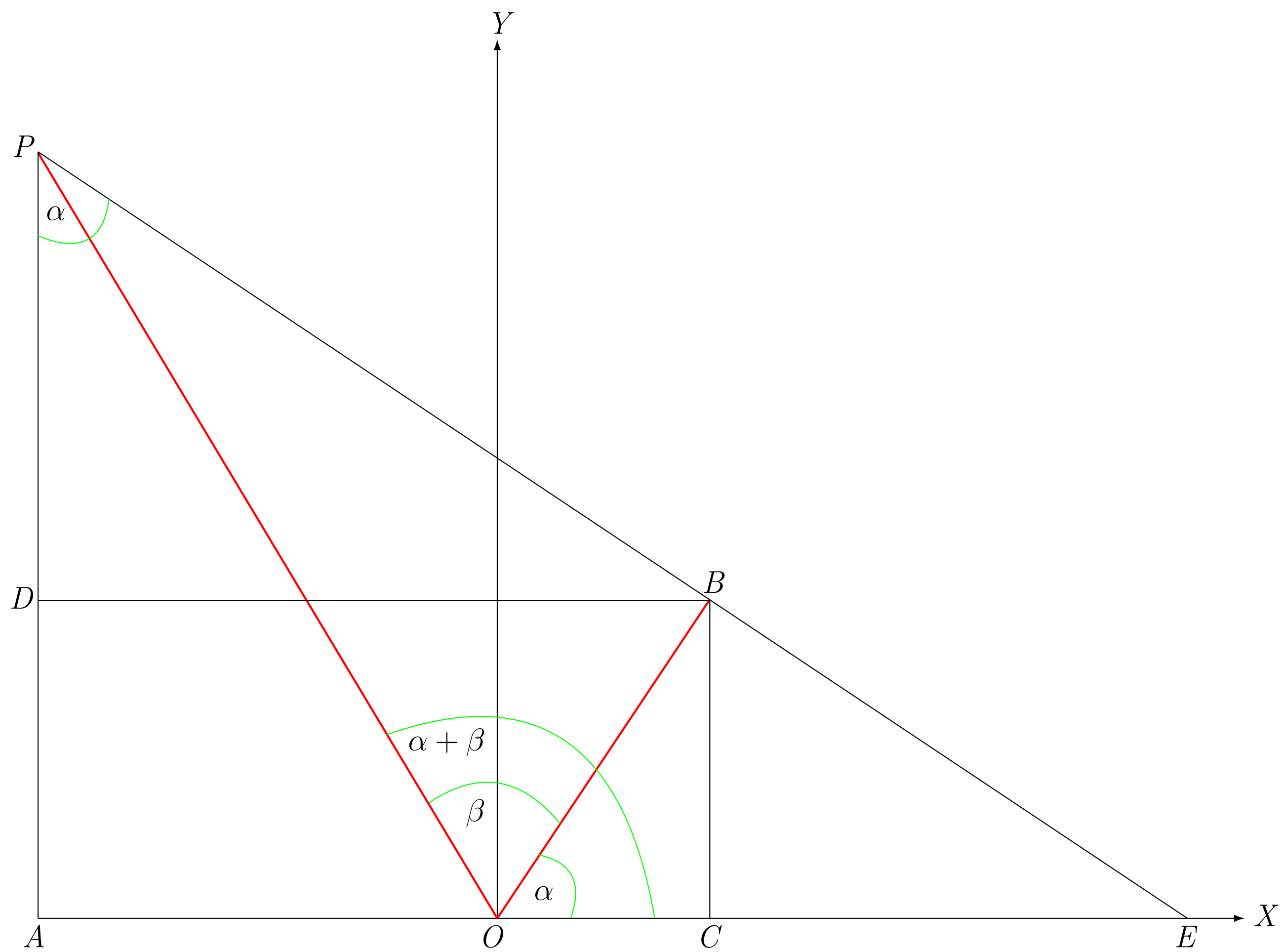

Figure 2: $90^\circ < \angle\alpha + \angle\beta < 180^\circ$

The figure for Case 2 looks different from the figure for Case 1 but is contstructed in essentially the same way. Note that line OB no longer intersects line PA and that point E now lies on line OX.

In this case we have to prove that $\angle\alpha = \angle APE$. We start by showing that $\triangle COB \sim \triangle CBE \sim \triangle APE$.

$\angle OBC + \angle CBE = 90^\circ$	By construction.
$\angle CEB + \angle CBE = 90^\circ$	Sum of acute angles in $\triangle ECB$.
$\angle OBC = \angle CEB$	

Triangles COB and CBE have two angles equal and are therefore similar. It is obvious from Figure 2 that $\triangle CBE \sim \triangle APE$. Then it follows that $\angle\alpha = \angle CBE = \angle APE$.

The algebra for proving the two formulas is then similar to that of Case 1.

Reference

Frank Ayres Jr. Schaum's Outline of Theory and Problems of Trigonometry. The McGraw-Hill Companies, Inc., New York, 3rd edition, 1999. See p. 102.