Rate of Change and Average Rate of Change

For a function $f(x)$, we introduced the *rate of change* over a (usually small) interval of length h, as $f(x+h)-f(x)$. Consequently, we defined the *average rate of change* over the same interval as $\frac{f(x+h)-f(x)}{h}$.

The book discusses how the average rate of change of a polynomial changes if we keep h fixed, and vary the point x. In the following discussion, h will be assumed to be small. Note that as soon as $h<1$, $h>h^2>h^3>\ldots$, with the inequalities becoming stronger the smaller h is.

For example, if $f(x)=2x^3+x^2-3x+1$, we have

$$\frac{1}{h}\left(2\left[x+h\right]^3+\left[x+h\right]^2-3\left[x+h\right]+1-2x^3-x^2+3x-1\right)=$$

$$=\frac{1}{h}\left(6x^2h+6xh^2+2h^3+2xh+h^2-3h\right)=$$

$$=6x^2+2x-3+(6x+1)h+2h^2$$

If h is very small, the average rate of change of our third degree polynomial is almost equal to a second degree polynomial (up to a term proportional to the small quantity h). As discussed in the book, this is a general fact: the average rate of change of a polynomial of degree n, for h fixed, is given by a polynomial of degree $n-1$ plus terms that are small if h is small (in general, there will be terms proportional to h, h^2, and on, up to h^{n-2}, which will be even smaller)

We can ask a different question: how does the *rate of change* depend on h, for a given value of x? Using the same example, we have that

$$f(x+h)-f(x)=6x^2h+6xh^2+2xh+h^2-3h=\left(6x^2+2x-3\right)h+\left(6x+1\right)h^2+2h^3 \tag{1}$$

We notice three terms (because we are working with a third degree polynomial). The first is proportional to h, and the coefficient is exactly the average rate of change. What about the second term, proportional to h^2 (hence smaller). We can check that $6x+1$ is half the average rate of change of $6x^2+2x-3$ (up to a small error, proportional to h):

$$\frac{1}{h}\left[6\left(x+h\right)^2+2(x+h)-3-6x^2-2x+3\right]=$$

$$=12x+6h+2=12x+2+6h=2\left(6x+1\right)+6h$$

This is actually only a step in a pattern: the third term has a coefficient, 2, which is equal to $\frac{1}{6} = \frac{1}{2 \cdot 3}$ times the the average rate of change of the average rate of change $12x + 2$ (up to the usual small error), that is its slope (it is linear) divided by 6.

If we had worked with a higher order polynomial, we could have gone further, and if you try it, you will see that a fourth term (in h^4) would have a coefficient almost equal to the average rate of change of the previous term, divided by $\frac{1}{2 \cdot 3 \cdot 4} = \frac{1}{24}$, and so on.

While at this point all of this is just a curiosity, it turns out to be a basic feature of what is known as *differential calculus.*

However, a formula like (1) gives significant information. Indeed, it tells us that if h is small enough that we can ignore the terms in h^2 and h^3, the graph of f near x will coincide almost exactly with that of the straight line $\left(6x^2 + 2x - 3\right) h$ (remember that here x is fixed, and we look at this as a function of h, that is we are looking at the values of f near $f(x)$). That will be the tangent line to f at x, $6x^2 + 2x - 3$ being its slope! But what if x is such that $6x^2 + 2x - 3 = 0$? Then we cannot ignore the term in h^2any more, since it is the largest term, and the graph will be close to that of the quadratic function $(6x + 1)h^2$near $h = 0$, that is it will exhibit a maximum if $6x + 1 < 0$ and a minimum if $6x + 1 > 0$ at x!

Even if the term proportional to h does not vanish, if h is such that we can only ignore h^3compared to h and h^2, the formula tells us that the graph of f near x will not be quite a straight line, but almost coincide with the graph of the quadratic function $\left(6x^2 + 2x - 3\right) h + (6x + 1) h^2$. $6x+1$ being the coefficient of the quadratic term, it will tell us that the graph will look like that of a concave up parabola if $6x + 1 > 0$ near x, of a concave down parabola if $6x + 1 < 0$. What of $6x + 1 = 0$? Now the h^3term cannot be ignored any more, so the graph will look like that of a cubic function, where the concavity changes (this is called an *inflection point*). While our example *is* a cubic function, so this last "approximation" is actually exact, this argument works perfectly well for a higher order polynomial which would produce also terms in $h^4$and higher.