Quadratic Functions and Equations

1 Reminder on Quadratic Equations

Quadratic equations are equations where the unknown appears raised to second power, and, possibly to power 1. After simplifications, equations all reduce to the form

$$ax^2 + bx + c = 0$$

and the solutions are (assuming $b^2 - 4ac \geq 0$)

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac < 0$ there are no real solutions.

Note If you wish, in the last case, you could talk of "complex solutions" $-\frac{b}{2a} \pm i\frac{\sqrt{4ac-b^2}}{2a}$, but when this comes as solution to a problem leading to a quadratic (or leading to a quadratic – see below) equation, fact is the problem has no solution (typical example: intersection between a line and a circle that do no meet). Complex number were first introduced when such "ghost numbers" turned out to work as intermediate steps where the "imaginary" parts eventually canceled out, in the solution of cubic equations. From there, the theory expanded into a major beautiful area of math, but within our context, they would not serve any purpose. They are very useful in more advanced areas, but, for that, their study has to go well beyond the simple attachment of a symbol to $\sqrt{-1}$.

In special cases, you can find the solution much faster in a direct way, rather than by using the formula: e.g.

$$ax^2 = b$$

$$x = \pm\sqrt{\frac{b}{a}}$$

or

$$ax^2 + bx = 0$$

$$x = 0 \quad x = -\frac{b}{a}$$

Examples:

1. $2x^2 + 3x - 5 = 0$:

$$x = \frac{-3 \pm \sqrt{9+40}}{4} = \frac{-3 \pm 7}{4}$$

$$x = 1 \quad x = -\frac{10}{4} = -\frac{5}{2}$$

2. $6x^2 - 1 = 0$

$$6x^2 = 1$$

$$x^2 = \frac{1}{6}$$

$$x = \pm\frac{1}{\sqrt{6}} = \pm\frac{\sqrt{6}}{6}$$

2 Equations That Lead to Quadratic Equations

Sometimes, though the equation is not quadratic, simple manipulations will lead to a quadratic equation whose solutions are (or *may be*) solutions to our original problem, or that will lead to the sought for solutions after a simple additional step. The cautionary note reminds us that some manipulations can lead to the appearance of "spurious" solutions, i.e., solutions of the modified problem that are not solutions to the original.

Examples of problems of this type are:

1. $\sqrt{x^2 - 1} = 1$ (squaring the two sides leads to a quadratic equation)

2. $\frac{2}{-5x^2+2x+1} = 2$ (multiplying both sides by the denominator of the left hand side leads to a quadratic equation)

3. $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 2 = 0$ (solving for the new unknown $y = x^{\frac{1}{3}}$ leads to a quadratic equation in y. We will find the solutions we need, by taking cubes of the solutions: $x = y^3$).

4. $2x^8 + 5x^4 - 1 = 0$ (solving for the new unknown $y = x^4$ leads to a quadratic equation in y

$$2y^2 + 5y - 1 \tag{1}$$

We will find the solutions we need, by taking fourth roots of the solutions: $x = \pm y^{\frac{1}{4}}$ - assuming the solutions are positive).

Note that example 2 is a *rational equation*, and that we need to make sure that the solutions to the quadratic equation this leads to are not such as to make the denominator take the value 0 too! Here we would have

$$\frac{2}{-5x^2+2x+1}\cdot\left(-5x^2+2x+1\right)=2\left(-5x^2+2x+1\right)$$

$$2=-10x^2+4x+2$$

$$0=-10x^2+4x$$

This is easily solved by noticing that $-10x^2+4x=x\,(-10x+4)$, so that to be zero we need either $x=0$ or $-10x+4=0$, i.e. $x=\frac{4}{10}=\frac{2}{5}$. We can then check that neither of these two values will cause the denominator to be zero. As for example 4, we can quickly see that equation (1) has two real solutions, one positive, and the other negative. For our purposes, only the positive solution is useful. Hence we take

$$y=\frac{-5+\sqrt{25+8}}{4}=\frac{-5+\sqrt{32}}{4}=\frac{-5+4\sqrt{2}}{4}=-\frac{5}{4}+\sqrt{2}$$

(just several equivalent expressions for the positive root), and the solutions to our original equation are

$$\pm\sqrt[4]{\sqrt{2}-\frac{5}{4}}$$

3 Getting Information About Quadratic Functions

Quadratic functions are especially easy to describe: if $y=f(x)=ax^2+bx+c$, then

- the y-intercept is $y=c$
- the x-intercepts are two if $b^2-4ac>0$: $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
- they reduce to one if $b^2-4ac=0$: $-\frac{b^2}{2a}$
- there are none if $b^2-4ac<0$
- if $a>0$ the function has a minimum for $x=-\frac{b}{2a}$, and the minimum value is $-\frac{b^2}{4a}+c=\frac{4ac-b^2}{4a}$
- if $a<0$ the function has a maximum for $x=-\frac{b}{2a}$, and the maximum value is $-\frac{b^2}{4a}+c=\frac{4ac-b^2}{4a}$

Examples:

1. $y=f(x)=2x^2-x-10$: $b^2-4ac=(-1)^2-4\cdot2\cdot(-10)=1+80=81>0$. the y-intercept is $y=-10$, the x-interceptors are at $\frac{1\pm\sqrt{81}}{4}$, i.e,. $x=\frac{10}{4}=\frac{5}{2}$, and $x=-\frac{8}{4}=-2$. The function has a minimum at $x=\frac{1}{4}$, where its value is $-\frac{1}{8}-10=-\frac{81}{8}$

2. $y = f(x) = -3x^2 + 2$: We don't need the formula to find that there are two x-intercepts, $x = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$. The y-intercept is $f(0) = 2$, and the minimum (the graph is concave down) is at $x = 0$, where $y = 2$.

4 Applications of Quadratic Equations and Functions

4.1 Examples of Economy/Business Applications

4.1.1 Revenue Models

Suppose we sell an item at price p. Suppose also, that we are given a "supply-demand" function, of the form $D(p) = a - bp$, asserting that we will sell $D(p)$ items, if the price is p.

We now want to evaluate what our revenue will be, if we set the price at p. Since we are selling $D(p)$ items, and each fetches a price of p, we will have a revenue given by

$$R(p) = pD(p) = p\,(a - bp) = ap - bp^2$$

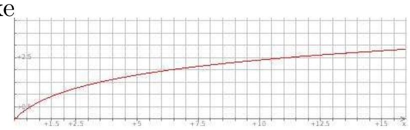
Since $R(p)$ is a quadratic function, and since $a > 0$, and $b > 0$, its graph will be open down, increasing at $p = 0$, and it will thus have a positive maximum - a value of p, at which earnings per item and sales balance out to provide maximum revenue.

You will also notice other easy facts, like $p = 0$ implies $R(p) = 0$ (obviously: giving stuff for free produces no revenue), as well as $R(p) = 0$ if $D(p) = 0$ (obviously, no sales, no revenue).

4.1.2 Quadratic Utility Functions

When you are a billionaire, an extra \$100 won't mean much to you, but if you are broke, it makes a whale of a difference. To express the "diminishing marginal value of money" (i.e., the more you have, the less a given fixed amount of money will mean to you), economists have long tried to use the concept of "utility".

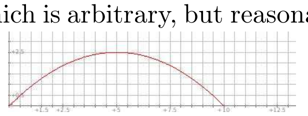
The idea is this: let x be your wealth. As x increases, the "utility" this money has for you increases, but at a slower rate, the larger your wealth. Something like



Now, to build quantitative models, we will have to choose a specific expression for a utility function. There is a whole cottage industry for that, and we will see more popular examples elsewhere (logarithmic functions are popular, for example), but here we can consider a choice that is often ridiculed, but also often put forward, if for no other reason, because of its simplicity: quadratic utility functions (open down, so that they will, at first at least, increase, but at an ever slower rate).

The problem with quadratic functions is that they don't always increase. Hence, we need to *restrict the domain* of a quadratic function that should serve as utility (presumably, it doesn't happen that people get so rich that they actually hate getting richer....).

For example, take the function $u(x) = x - 0.1x^2$ (we choose to set $u(0) = 0$, which is arbitrary, but reasonable - also, x is wealth, but in some arbitrary unit):



This graph peaks at $x = 5$, and if we don't want to believe in a *decreasing* utility function, we will restrict this function to $0 \leq x \leq 5$ (see section 5 for more about this idea of restricting functions).

4.2 Applications To Mechanics

We'll just look at two simple applications from mechanics.

4.2.1 Motion of a Falling Body

Classical Physics tells us that a body of any mass, falling in absence of friction, i.e., of air (remarkably, the weight of the body has no relevance for its motion in this case), will follow the following law: starting at time $t = 0$, at height $h = h_0$, and velocity $v = v_0$ ($v_0 > 0$ if the starting velocity is upward, $v_0 < 0$, if it is downward), will be, at time t, at a height given by

$$h(t) = h_0 + v_0 t - \frac{1}{2} g t^2$$

where g is a constant measuring the attraction exerted by the Earth.

A typical problem we can solve is "when will the body hit the ground". If we are measuring h so that the ground is at $h = 0$, this is easily solved by solving the quadratic equation

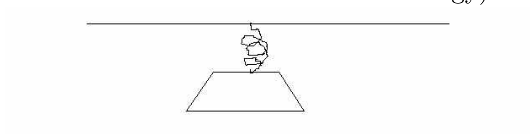
$$h_0 + v_0 t - \frac{1}{2} g t^2 = 0$$

You should graph an example, with specific numbers, and find out for yourself what to do with the "other" solution to this problem :) [in International Scientific Units, $g = 9.8 \quad m/s^2$]

4.2.2 Energy of a Spring

A spring exerts a "restoring" force when it is extended. For most cases, if the spring is elongated form its rest position by x, this force can be described by Hooke's Law, as $-kx$ - i.e., pushing in the opposite direction of the elongation x, and proportional to it (k is a constant depending on the spring). It turns out that at elongation x, the spring has an energy $\frac{1}{2}kx^2$.

Now, suppose we attach a weight to the spring, and let the weight pull the spring down from its rest position (starting at rest, the weight does not contribute additional initial "kinetic" energy):



The weight loses "potential energy" equal to gx as it drops a height of x. So, as there is an increase in the energy of the spring, there is a decrease in the energy of the weight. However, the two are not equal: one increases proportionally to x^2, and the other increases proportionally to x. The difference is transferred to the "kinetic" energy: the energy associated with the speed of the falling weight. The fall will reach its lowest point when there is no velocity any more - hence, no kinetic energy - or when the energy gained by the spring equals exactly the energy lost by the weight. This happens when x is the solution to the equation

$$\frac{1}{2}kx^2 = gx$$

a simple quadratic equation.

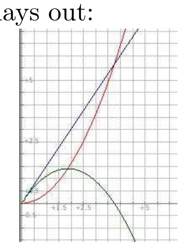
Solving this equation gives

$$x\left(\frac{k}{2}x - g\right) = 0$$

or $x = 0$ (that's the initial position, when there is no energy from the spring, nor additional energy from the vertical position), and

$$x = \frac{2g}{k}$$

Note that this height is twice the height at which the two opposing forces (the spring and gravity) are equal - that's when $g = kx$. Fact is, as the weight falls through this position, it will keep going, as it has kinetic energy to spend. Comparing the graphs of $y = \frac{1}{2}kx^2$, and $y = gx$ may give an idea of how this plays out:



The red curve is a parabola ($y = \frac{k}{2}x^2$), the blue curve is a linear function ($y = gx$), and the green curve is their difference - the largest difference, corresponding to the largest kinetic energy, i.e., the largest velocity, is halfway to the end of the excursion.

5 A Remark On Domains and Ranges in Word Problems

A word problem may restrict the domain of the functions it uses because of additional constraints due to the application.

Examples:

1. The demand function for an item, as a function of the price p, is $D(p) = 20 - 2p$. Find the domain of the corresponding revenue function $pD(p)$: we can't have $p < 0$, nor $D(p) < 0$, hence $p \geq 0$, and $20 - 2p \geq 0$, or $p \leq 10$. All in all, $0 \leq p \leq 10$

2. A body sent up in the air with initial velocity $v_0 = 10\text{m/s}$, will be at height $v_0 t - \frac{1}{2}gt^2$ after time t, where $g = 9.8m/s^2$. Since it cannot go below the ground, once it has fallen back, the domain of this function is restricted by $10t - \frac{9.8}{2}t^2 \geq 0$, i.e., $t \geq 0$, and $10 - 4.9t \geq 0$ or $0 \leq t \leq \frac{10}{4.9}$

3. The revenue we get from selling x widgets is $R(x) = 100x(x - 20)$. This implies that there will be no sales if $x < 20$. The cost to produce the same x widgets is $C(x) = 10 + \frac{x}{100}$. The profit from the sale of these widgets is $P(x) = R(x) - C(x)$. The domain of this function, constrained by the application, is thus $[20, \infty)$.