New Functions from Old

Once we have a library of familiar functions, we can use them as building blocks to construct new functions. We've already done a little of that with transformations – seeing what changes in the graph come from changing the constants that appear in the formula. Now we'll learn a few new ways to combine functions to make new functions.

Arithmetic

This is exactly what it sounds like. We can combine functions using the four operations of arithmetic. That is, we can add, subtract, multiply or divide functions.

Example: Let $f(x)=4x^2-13$ and $g(x)=\sqrt{x-2}$. Then their quotient, $\frac{f}{g}(x)$, is the function whose outputs are the quotients of the outputs from *f* and *g*. The algebraic rule for is simply the quotient of the two rules: $\frac{f}{g}(x)=\frac{f(x)}{g(x)}=\frac{4x^2-13}{\sqrt{x-2}}$.

If the component functions are given as tables or graphs, you can make a table or graph for the arithmetic combination by performing the arithmetic operation on the outputs. See, for example, Example 2 on p. 212 of the Precalculus (Stewart) textbook.
We have to knit together the domains of the component functions to find the domain of the combination. It is tempting to write the new function, whether as a formula, a graph, or a table, and then find the domain from scratch. This will usually work, except in some cases

Example: Let $f(x)=4x^2-13$ and $g(x)=\sqrt{x-2}$. Find the domain of $\frac{f}{g}(x)$.

Solution: We just figured out the formula above: $\frac{f}{g}(x)=\frac{4x^2-13}{\sqrt{x-2}}$. Now let's find the domain of this. There are no special conditions mentioned, and this isn't an applied problem, so we can focus strictly on the formula itself. We can't divide by zero, so we have to avoid using $x = 2$ as an input. We can't take the square root of a negative number, so we must also avoid $x < 2$. That's it – the domain is everything that's left.

The domain of $\frac{f}{g}(x)=\frac{4x^2-13}{\sqrt{x-2}}$ is the set of all real numbers greater than 2, or $(2, \infty)$.

Simple Counterexample: Let $f(x)=x^2-1$ and $g(x)=x+1$. Then, $\frac{f(x)}{g(x)}=\frac{x^2-1}{x+1}=\frac{(x+1)(x-1)}{x+1}=x-1$. So, it looks like the domain is given by all real numbers. However, *g* is undefined when $x=-1$, so the domain of the ratio has to exclude this point, since we cannot do the operation when the denominator is undefined!

Composition

One of the most important ways to combine functions is to chain them together, using the output from one as the input into another. A simple example of this is unit conversion – we have one function that tells us how many meters high the ball is after *t* seconds, and another that tells how many feet are in a certain number of meters. We can use the output of the first function (meters) as the input to the second function to find how many feet high the ball is after *t* seconds. The chaining together of functions in this way is called composition. The composition of *f* with *g*, written $f \circ g(x)$, is the function that takes *x*, first does *g* to it, and then does *f* to the output. That is, $f \circ g(x) = f(g(x))$.

Example: Let $f(x) = 4x^2 - 13$ and $g(x) = \sqrt{x-2}$. Then their composition

$$f \circ g(x) = f(g(x)) = f\left(\sqrt{x-2}\right) = 4\left(\sqrt{x-2}\right)^2 - 13. = 4(x-2) - 13 = 4x - 21$$

When it comes to find the domain, we need the first function to be defined (here, the domain of *g* is $[2, \infty)$), and also need its output to be in the domain of the second function. In our example, the domain of *f* is the whole real line, so that is not an issue, but, in general, it will be. In our case, the domain is $[2, \infty)$, and not the whole real line, which is what you may think if you only looked at the final formula.

Decomposition

Sometimes you will be given the composition and be asked to identify the component pieces. This is called "decomposition." It turns out to be a very useful skill in calculus. There are often several correct decompositions for a function, but usually only one of them is useful. It may take some practice before you can see which composition is the useful one.

Example: The function $G(x) = \dfrac{1}{x+3}$ is a composition $f \circ g$. Identify the component functions *f* and *g*.

One solution: Let $f(x) = x$, and $g(x) = \dfrac{1}{x+3}$. Then

$$F(x) = f(g(x)) = f\left(\frac{1}{x+3}\right) = \frac{1}{x+3}$$

. This is a correct answer. However, you can see that we didn't really get anywhere – one of our component pieces is just what we started with. In general, if you let either the inner function or the outer function be the identity functions (that is, simply $f(x) = x$), you can quickly write a correct decomposition. But it isn't useful. In this class, you want to look for the most useful solution. **Never use the identity function as either of your component pieces.**

The most useful solution: In many cases, there is an "obvious" choice, which you can find by thinking about the inside and the outside. In $G(x)=\frac{1}{x+3}$, the "inside function" is the denominator and the "outside function" is the reciprocal function (that is, "one over"). In the composition $f \circ g$, g is the inside function and f is the outside function. So this decomposition would be $f(x)=\frac{1}{x}$ and $g(x)=x+3$. Then

$$F(x)=f(g(x))=f(x+3)=\frac{1}{x+3}.$$

Another correct solution: Here's another one that works: Let $g(x)=2x+6$ and $f(x)=\frac{2}{x}$. Then $F(x)=f(g(x))=f(2x+6)=\frac{2}{2x+6}=\frac{1}{x+3}$. This decomposition is correct, but it seems strange and, crucially, not providing any additional insight over the previous one.

There are usually lots of correct solutions, some of which involve some creativity to find. In this class, you won't need to find any of these "clever" decompositions, but if you do find one, it will be correct. Whether such a "clever" decomposition is worth it, however, depends on the context, that is, on what use it will put to. Unfortunately, if the problem was simply to find a decomposition, there is no context, and there is no real way to choose one over another (except if on of the two is trivial, as discussed above).