Additional Notes on "New Functions From Old"

Composition of Functions

1 Arithmetic

As the note says, when looking for the domain of a function built from others, the simplest mean is, usually, to write out the result and get the domain directly. So, for example, if

$$f(x) = x^2 + 1 \quad g(x) = \sqrt{1-x}$$

the domain of $\frac{f}{g}$ is most easily found by noting that

$$\frac{f(x)}{g(x)} = \frac{1+x^2}{\sqrt{1-x}}$$

and we need both

- $\sqrt{1-x} \neq 0$, i.e. $x \neq 1$, and
- $1 - x \geq 0$, i.e. $x \leq 1$

which, combined, give us $x < 1$ as the condition.

There is, however, a subtle "gotcha" here... In fact, think formally. If you are building a new function from, e.g., $\frac{f}{g}$, its domain must be such that both f and g can be evaluated, and, additionally, we need $g \neq 0$. Using this method for our first example, we still get the same result: in fact, f is defined everywhere. g needs $x \leq 1$ to be defined, and the requirement $g \neq 0$ implies $x \neq 1$. Let's take a different example, though. Suppose

$$f(x) = x(1-x) \quad g(x) = 1 - x^2$$

If we go by the last rule, we see that both functions are defined everywhere, but $g \neq 0$ requires $x^2 \neq 1$, or $x \neq \pm 1$. Let's now build the function $\frac{f}{g}$:

$$\frac{f(x)}{g(x)} = \frac{x(1-x)}{1-x^2} = \frac{x(1-x)}{(1-x)(1+x)} = \frac{x}{1+x}$$

and, all of a sudden, we find ourselves a larger domain: $x = 1$ causes problems no more... How come? Well, we did say that $x \neq 1$ was a condition. To arrive at the final formula, we divided both numerator, and denominator by $1 - x$.

If we did that when $x = 1$, we would have divided by zero - a huge no-no (dividing by zero is meaningless, but if you insist on doing it, I'll be happy to prove to you that $0 = 1$). So, what is the domain of $\frac{f}{g}$ in this case? As a purist mathematician, I would answer $x \neq \pm 1$. If you answered $x \neq -1$ only, I would have to comment your answer with this long discussion, and would not penalize you much (or at all...). To get out of this quandary cleanly, try to take the first calculus class as soon as possible :-)

2 Composition

The previous section illustrated how there's subtle things can that happen to the domain of a function built from other functions. With composition, we can build even more interesting examples.

Here's a typical delicate point:

$$f(x) = x^2 \quad g(x) = \sqrt{1-x}$$

What is the domain of $f \circ g$? Using the precise definition of composition, we would take an x, and compute $g(x)$. This can be done only if $x \leq 1$. Now, we can compute $f(g(x))$. Hence, the domain is $\{x \mid x \leq 1\}$.

If we work out a formula for $f \circ g$, however, we find

$$f(g(x)) = \left(\sqrt{1-x}\right)^2 = 1 - x$$

and the domain is now all real numbers,

How is that? Well, the problem is that $f \circ g$, and $1 - x$ coincide, but only when they are both defined - and $f \circ g$ is not defined for $x \leq 1$

Remark Sure, we could eliminate this particular unpleasantness by introducing complex numbers, so that now $\sqrt{1-x}$ is defined for all x. But that, really would create a lot of new issues. In fact, we would be trying to work with *complex functions of a complex variable*, and it turns out that they are a very peculiar breed - different in so many ways form their real cousins. Not coincidentally, if you were interested in them, you would have to take a quite advanced college course.... And, of course, one can be even more creative in building these type of examples.