Irrational Values

Ever since the ancient Greeks realized that $\sqrt{2}$ (the length of the diagonal to a square of side 1) cannot be expressed as a fraction, we know that as soon as we leave the comfort of the four elementary operations (sums/differences and products/quotients), we end up almost always with irrational values. That's a fact, as we can check right now. We only consider positive numbers, since none of the following operations can be applied to negative numbers (and, for logarithms, to zero).

1 Roots of rational numbers

Consider the $r-$th root ($r \geq 2$) of the number $\frac{m}{n}$, where the fraction has been reduced to lowest terms (if the number is an integer, then $n = 1$). Suppose this root was rational, say, $\frac{p}{q}$, also reduced to lowest terms. This means that

$$\left(\frac{p}{q}\right)^r = \frac{p^r}{q^r} = \frac{m}{n}$$

or

$$np^r = mq^r$$

These are now two equal integers, and so they have the same prime factors. Since p and q have no common factors, just as m and n haven't, m must be a factor of p^r, which then is equal to mk for some integer k, and n must be a factor of $q^r = nh$ for some integer h. So,

$$mnk = mnh$$

or $k = h$, but since p and q have no common (non trivial) factor, we have $h = k = 1$, and $m = p^r$, $n = q^r$, so the fraction $\frac{m}{n}$ is actually an exact power. All fractions that are not exact powers have irrational roots.

2 Decimal Logarithms of Rational Numbers

In this and the next section we assume that the number we are taking the log of is greater than 1. If $\frac{m}{n} < 1$, we just apply the argument to $\frac{n}{m}$.

Suppose the rational number $\frac{m}{n}$ (again, reduced to lowest terms) has a rational decimal logarithm, say, $\frac{p}{q}$ (again, lowest terms). This means

$$10^{p/q} = \frac{m}{n}$$

or

$$10^p = \frac{m^q}{n^q}$$

$$n^q 10^p = m^q$$

Now, again, we have two integers and their prime factors have to be the same, but m and n have no common factors (hence so do m^q and n^q), so this can only happen if $n = 1$ and then $m^q = 10^p$, and a moment's reflection shows that this can only happen if m is a power of 10 (recall that $m = 10^{p/q}$ cannot happen if m is an integer and p/q is not an integer: m would be the $q - th$ non exact root of the integer 10^p, and we know from the previous section that this would be an irrational number).

3 Natural Logarithms of Rational Numbers

This is a somewhat elegant argument, as it is based on the (deep) fact that e (just like, for example, π) is a *transcendental number,* that is, it is not the root of any polynomial with integer coefficients. Now if the rational number $\frac{m}{n}$[1] had a rational natural logarithm, say, $\frac{p}{q}$, we would have

$$e^{p/q} = \frac{m}{n}$$

$$e^p = \frac{m^q}{n^q}$$

$$n^q e^p - m^q = 0$$

and e would be a root of the polynomial $n^q x^p - m^q$, which is impossible (recall that $p \geq 1$).

[1] We exclude $\frac{m}{n} = 1$, which has natural logarithm 0, so, in the following, $p \neq 0$, and $m \neq n$