A note on the irrationality of natural logarithms

We take it for granted that natural logarithms of integer numbers are irrational. Actually, it is a fact, which means that almost all such logarithms can only be written out as approximate numbers. That's actually easy to see, and it is similar to the proof that non exact roots of integers are irrational.

Roots If the integer p is not equal to n^r for some natural numbers n, and $r \neq 1$, the $p^{\frac{1}{r}}$ is irrational. Suppose it was not. Then, for an irreducible fraction $\frac{a}{b}$, $p = \left(\frac{a}{b}\right)^r$. That means that $pb^r = a^r$. So a^r has to have p as a factor, which forces it to have p^r as a factor, so $a^r = p^r q$ for some q, prime with p. Now, $pb^r = p^r q$, so $b^r = p^{r-1}q$, and b also has p as a factor, contrary to the assumption that it had no common factor with a. This is a variation on a very well known proof about the irrationality of $\sqrt{2}$, which supposedly dates back to classical Greece: if $2 = \frac{a^2}{b^2}$, where a and b have no common factors, then $2b^2 = a^2$, so a is even, and equal to $2k$, so $a^2 = 4k^2$, and $b^2 = 2k^2$, so b is also even, and has a common factor, 2, with a, contrary to our assumption. Since rational numbers are ratios of integers, and for an irreducible fraction $\frac{p}{q}$ to be equal to $\left(\frac{a}{b}\right)^r$, where $\frac{a}{b}$ is also irreducible, we need $p = a^r, q = b^r$, the statement about irrationality of roots extends to rational numbers.

Logarithms For decimal logarithms it is easy to see that, unless the natural number $n = 10^k$ for some natural number k, $\log n$ is irrational. If it wasn't, $n = 10^{\frac{p}{q}}$, for an irreducible fraction $\frac{p}{q}$. But 10^p is an integer and is not the q-th power of another integer, so by the preceding argument, its q-th root cannot be an integer, and is irrational. As for natural logarithms, we have to rely on a deep result: the number e (just as π) is not only irrational, it is a *transcendental irrational* number. That means that is irrational, but also not the solution of any algebraic[1] ("polynomial with integer coefficients = 0") equation – in particular, it is not the root of any rational number. Now, suppose the irreducible fraction $\frac{p}{q}$ had a rational natural logarithm $\frac{a}{b}$. Then, $e^{\frac{a}{b}} = \frac{p}{q}$, that is,

$$\left(\frac{p}{q}\right)^b = e^a$$

The left hand side, being an integer power of a rational number is rational, but the right hand side, being and integer power of a transcendental irrational number cannot be rational.

[1] We could also say "polynomial with rational coefficients", with no gain in generality, since any equation $p(x) = 0$, where p is a polynomial with rational coefficients is equivalent the equation $Lp(x) = 0$, where L is the least common denominator of the coefficients, which has integer coefficients.