Inverse Functions

The word "inverse" means backwards, and that's what inverse functions are about – going backwards. There are a few different and useful ways to think about inverse functions.

Swapping the roles of input and output

One important reason we care about inverse functions is that, in many cases, the same relationship can give two different functions, depending on what questions you're interested in answering. Which function you use depends on which quantity you want to use as your input.

Example: A private investigator charges a $500 fee per case, plus $80 per hour that she works on the case. There is a functional relationship between the hours she works and the amount she bills. But which is the input and which is the output?

If the number of hours she works is the input, then the number of dollars she bills is the output. And it's a function, because each possible number of hours is associated with exactly one billing amount. This might be the function you'd think of first. If we let *h* be the number of hours the detective works and *b* be the number of dollars she bills, then this function might be written as

$$b = f(h) = 500 + 80h \quad .$$

You'd use this function if you knew how many hours she worked on your case and you wanted to know how much she would charge you.

But the very same relationship yields a different function, whose input is the billing amount and whose output is the number of hours she works. This is also a function, because each possible bill is associated with exactly one amount of time. Again, letting *h* be the number of hours and *b* be the amount she bills in dollars, we can write this function:

$$h = f^{-1}(b) = \frac{b - 500}{80}$$

This would be a helpful function if you had a certain amount of money to spend and you wanted to know how many hours she would work on your case.

The two functions here are inverse functions. They model the same relationship, but the roles of input and output have been exchanged. That little –1 that looks like an exponent for the *f* in the second formula indicates it is the inverse function for *f*. (**It is not an exponent.**)

Undoing

The most important reason we want to study inverse functions is that they undo each other. This will be useful when we solve equations involving functions. The fact that inverse functions undo each other is where the definition of inverse functions comes from:

$f(x)$ and $f^{-1}(x)$ are inverse functions means that their composition in either order is the identity function. That is, both

$$f\left(f^{-1}(x)\right)=x \quad \text{and} \quad f^{-1}\left(f(x)\right)=x$$

The arrow diagram may be clearer:

$$x \to f^{-1}(x) \to f\left(f^{-1}(x)\right) \quad \text{and} \quad x \to f(x) \to f^{-1}\left(f(x)\right)$$

These are the algebraic conditions you must check in order to be sure you have inverse functions. They say that *f* and f^{-1} undo each other.

Example: Show that the two functions $f(x)=3x+4$ and $g(x)=\frac{x-4}{3}$ are inverse functions.

Solution: We have to confirm that both compositions work out to be simply *x*:

$$f\left(g(x)\right)=f\left(\frac{x-4}{3}\right)=3\left(\frac{x-4}{3}\right)+4=(x-4)+4=x$$

$$g\left(f(x)\right)=g(3x+4)=\frac{(3x+4)-4}{3}=\frac{3x}{x}=x$$

Indeed, they do, so these functions are inverse functions.

Notice how these functions are related to solving equations: if I want to solve the equation $y=f(x)=3x+4$ for *x*, first I would subtract 4 from both sides of the equation, then I would divide by 3:

$$y=3x+4$$
$$y-4=3x$$
$$\frac{y-4}{3}=x$$

But notice that those are exactly the steps I take if I use *g* – the rule for $g(x)=\frac{x-4}{3}$ says "first subtract 4, then divide by 3." The formula for *g* gives the instructions for solving an equation involving *f*.

Finding the Inverse Function

In algebra class, you spent a lot of time learning how to solve equations. In this class, we want to think about everything in terms of functions. Inverse functions are the way to bring solving equations into function-land. Inverse functions give the instructions for solving an equation. And that's exactly the method for finding the inverse function for a function given as a formula. The method here assumes x is the name of the input and y is the name of the output; this is for convenience only, of course.

To find the inverse function for a function $y = f(x)$, solve the formula for x, the input variable. That will give you the inverse function written in an unconventional way:

$x = f^{-1}(y)$ (Remember that the inverse function expresses the same relationship, but the roles of input and output have been swapped.)

Depending on the setting, you may want to rewrite the formula so that x is the name of the input variable and y is the name of the output variable. This step, called "exchanging the variables" in the book, is not always appropriate. In fact, it is usually not appropriate in applied problems, where the inputs and outputs have specific meanings.

Example: Find the inverse function for $b = f(h) = 500 + 80h$, the function that tells how many dollars our detective will bill for h hours of work.

Solution: We want to solve $b = 500 + 80h$ for h:

$$b = 500 + 80h$$
$$b - 500 = 80h$$
$$\frac{b-500}{80} = h$$

And here's our formula: $h = f^{-1}(b) = \dfrac{b-500}{80}$. The input variable is now b and the output variable is h.

This makes sense, since this is the formula that takes the amount of the bill in dollars and gives the number of hours the detective works. This is an example where it would not make sense to exchange the name of the variables.

Example: Find the inverse function $f^{-1}(x)$ for $f(x) = x^3 - 1$.

Solution: Solve $y = x^3 - 1$ for x:

$$y = x^3 - 1$$
$$y + 1 = x^3$$
$$\sqrt[3]{y+1} = x$$

This gives our formula $f^{-1}(y) = \sqrt[3]{y+1}$, which is in fact the inverse function using y as the input variable.

However, in this problem we were asked to find $f^{-1}(x)$; that is, we were asked to use x as the input variable. So in this problem, we will exchange the variables to get $f^{-1}(x)=\sqrt[3]{x+1}$.

Graphically

If you graph a function and its inverse on the same axes, the inverse will be a reflection of the original across the line $y = x$. That's because the inverse function swaps the roles of input and output. On a graph, that means interchanging the order of the coordinates for every point. That is, if (x, y) is on the graph of $y=f(x)$, then (y, x) will be on the graph of $y=f^{-1}(x)$.

How do we know there is an inverse?

There is always an inverse relationship. But that inverse might not be a function. Since we really want to concentrate on functions in this class, we want to know when the inverse will be a function.

Inverse functions go backwards – they use the function outputs as inputs, and give the function inputs as outputs. In order for the inverse to be a function, it needs to be able to go back perfectly. That is, there needs to be exactly one (original) input for every (original) output.

A function is **one-to-one** if each output comes from exactly one input. In order for its inverse to be a function, a function must be one-to-one.

How can we tell from the graph of a function if it's one to one (so its inverse is a function)? The **horizontal line test** will tell us.

> If no horizontal line crosses the graph of a function more than once, the function is one-to-one and its inverse is a function.

Thus, one to tell if a function is one-to-one is to graph it and see whether it passes the horizontal line test. The algebraic way to tell if $f(x)$ has an inverse is to write an equation $y=f(x)$ and solve for x: f has an inverse if the equation has (at most) one solution for every choice of y.