Annuities and Loan Repayments

1 Discrete Compounding

Compound Interest

Recall briefly how compound interest, compounded at n equally spaced times in a year, works. Simple interest at rate r increases an investment of P dollars (the *principal*) by adding rP dollars to it over a given amount of time – usually this being a year. So after t years (t need not be an integer), the investment will be worth $P + rP \cdot t = P(1 + rt)$, a *linear* growth.

Compound interest works like this: after a certain amount of time, usually a fraction $\frac{1}{n}$ of a year (so, yearly if $n = 1$, every six months if $n = 2$, every month if $n = 12$, every day if $n = 365$, and so on) the interest is merged into the principal, and from then on interest is calculated on this larger amount. So, after one *compounding period* the principal will have grown to $\left(P + r\frac{1}{n}P\right) = P\left(1 + \frac{r}{n}\right)$. Now interest is calculated on this larger amount, so that after two compounding periods we are at $P\left(1 + \frac{r}{n}\right) + \frac{r}{n}P\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2$. This goes on, so that after m compounding periods we are at $P\left(1 + \frac{r}{n}\right)^m$

Annuities

An annuity consists in a periodic investment of a fixed amount of money at regular intervals. Interest is earned on the total amount deposited, at a rate of $i = \frac{r}{n}$, where r is the annual interest rate, and n the number of deposit times in a year.

Thus, starting from 0, with a regular payment of p dollars, we will have after one period $p(1 + i)$. We now make a second deposit of p,and at the end of the second period we will have accrued interest on this second deposit, as well as on $p(1 + i)$ accumulated in the first period, for a total of $p(1 + i) + p(1 + i)^2$. We repeat now with a third deposit, and we quickly realize that after m deposits we will have

$$p(1 + i) + p(1 + i)^2 + p(1 + i)^3 + \ldots + p(1 + i)^m = p\sum_{k=1}^{m}(1 + i)^k \qquad (1)$$

(using the convenient *sigma-notation* for long sums: $\sum_{k=1}^{m} a_k \equiv a_1 + a_2 + \ldots + a_k$).

The sum in (1) has a special form, and it can be summed explicitly as we can see now: if we are summing increasing powers $q, q^2, q^3, \ldots$ we can observe that $(1-q)\left(1+q+q^2+q^3+\ldots+q^m\right) = 1-q^{m+1}$, since distributing the first parenthesis in the second causes all terms to cancel except the first and the last. Hence

$$\sum_{k=1}^{m} q^k = q\left(1+q^2+\ldots+q^{m-1}\right) = q\frac{1-q^m}{1-q} \tag{2}$$

. Hence, the sum in (1) adds up to

$$p(1+i)\frac{1-(1+i)^{m-1}}{1-(1+i)} = p\frac{1+i-(1+i)^m}{-i} = \frac{p}{i}\left[(1+i)^m-(1+i)\right] =$$

$$= \frac{p}{r/n}\left[\left(1+\frac{r}{n}\right)^m - \left(1+\frac{r}{n}\right)\right]$$

Note that after time t (in years), you will have made $m = nt$ payments (actually, we should write $\lfloor nt \rfloor$, the largest integer less than or equal to nt, but we'll just remember without the special notation), and we can also write

$$p\frac{\left(1+\frac{r}{n}\right)^{nt} - \left(1+\frac{r}{n}\right)}{\left(\frac{r}{n}\right)}$$

Loan Repayment

A similar argument allows us to work out loan repayments. Suppose you take out a loan L at an annual interest rate of r, making payments n times a year, which is also the compounding period. You start with a debt of L. After one period, you will have a debt of $L\left(1+\frac{r}{n}\right) - p$. You payment earns interest too, so after two periods you will owe $L\left(1+\frac{r}{n}\right)^2 - p\left(1+\frac{r}{n}\right) - p$ (you made a second payment). Repeating, it is easy to see that after m periods you will owe

$$L\left(1+\frac{r}{n}\right)^m - p\sum_{k=0}^{m-1}\left(1+\frac{r}{n}\right)^k$$

Using again our formula (2), this becomes

$$L\left(1+\frac{r}{n}\right)^m - p\frac{1-\left(1+\frac{r}{n}\right)^m}{1-\left(1+\frac{r}{n}\right)} = L\left(1+\frac{r}{n}\right)^m + \frac{p}{r/n} - p\frac{\left(1+\frac{r}{n}\right)^m}{r/n} =$$

$$= \left(L - \frac{p}{\left(\frac{r}{n}\right)}\right)\left(1+\frac{r}{n}\right)^m + \frac{p}{\left(\frac{r}{n}\right)} \tag{3}$$

For your debt to decrease, it is now clear that you payment must be such that $L - \frac{p}{\left(\frac{r}{n}\right)} < 0$, or

$$p > L \cdot \frac{r}{n}$$

that is must exceed the interest that your debt is accumulating over each payment period – which should not be surprising. The loan will be repaid after $m = nt$ periods such that

$$\left(L - \frac{p}{\left(\frac{r}{n}\right)}\right)\left(1 + \frac{r}{n}\right)^{nt} + \frac{p}{\left(\frac{r}{n}\right)} = 0 \tag{4}$$

Using (4) we can answer questions about loan payment schedule. Given all but one of the quantities $L, p, t = \frac{m}{n}, r$ we can solve for the remaining one (given a loan, the interest rate, and the repayment horizon t, what should be my payment? Given a payment, the interest rate, and a repayment horizon, how much can I borrow, and so on). All these equations are easy to solve, except when the unknown is the interest rate, because that leads to an algebraic equation of degree $m + 1$, which has to be solved numerically.

2 Continuous Compounding

Continuous Compound Interest

We have seen that if a yearly interest rate of r is compounded n times a year, after m compounding periods, we will have a capital of $P\left(1 + \frac{r}{n}\right)^m$. After t years (t is not necessarily an integer), we will have had $m = \lfloor nt \rfloor$ compounding periods (the symbol $\lfloor nt \rfloor$ stands for "integer part", or "floor" of nt, and is the largest integer less than or equal to nt). Let's rewrite the formula $P\left(1 + \frac{r}{n}\right)^{\lfloor nt \rfloor}$ as follows

$$P\left(1 + \frac{1}{\left(\frac{n}{r}\right)}\right)^{r\frac{\lfloor nt \rfloor}{r}}$$

We now let $n \to \infty$, that is we compound at shorter an shorter intervals, down to "infinitesimally small" intervals. It is intuitive that with n tending to infinity we may ignore the "integer part", and rewrite the formula without it:

$$P\left(1 + \frac{1}{\left(\frac{n}{r}\right)}\right)^{r\frac{nt}{r}} = P\left[\left(1 + \frac{1}{\left(\frac{n}{r}\right)}\right)^{\frac{n}{r}}\right]^{rt} \tag{5}$$

Since r is fixed, calling $\frac{n}{r} = x$, we have $x \to \infty$. The limit

$$\lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x$$

is a remarkable limit.

The following are facts of varying difficulty to prove (from fairly easy to much harder) about this limit:

- For $x \geq 1$, $2 \leq \left(1+\frac{1}{x}\right)^x < 3$
- As x increases, $\left(1+\frac{1}{x}\right)^x$ increases
- Since, by the first two statements, $\left(1+\frac{1}{x}\right)^x$ is always increasing, but is bounded above by 3, it has to tend to a limit, which we call e (most likely in honor of the great mathematician Leonard Euler).
- It turns out that e is an irrational number, with approximate value 2.71828182845905

We conclude that the function (5) tends to a limit, namely Pe^{rt}, the formula for continuous compounding interest.

Annuities and Loan Repayments

If interest is compounded, we can adapt the formulas for annuities and loan repayments by taking the limit as $n \to \infty$ to describe what happens with *continuous compounding* (that is, interest is compounded instantaneously at any time). Note that the payments p mentioned in the formulas are *payments per period*. We can think of them as being equal to $\frac{1}{n}$ times a yearly payment $\mathcal{P}$, much like $\frac{r}{n}$ is the interest over a compounding period, $\frac{1}{n}$ times the yearly interest r. Substituting $p = \frac{\mathcal{P}}{n}$ in the formulas we have the following limits

2.1 Annuities

$p\frac{\left(1+\frac{r}{n}\right)^{nt}-\left(1+\frac{r}{n}\right)}{\left(\frac{r}{n}\right)} = \frac{\mathcal{P}}{n}\frac{\left(1+\frac{r}{n}\right)^{nt}-\left(1+\frac{r}{n}\right)}{\left(\frac{r}{n}\right)} = \frac{\mathcal{P}}{r}\left[\left(1+\frac{r}{n}\right)^{nt}-\left(1+\frac{r}{n}\right)\right]$. As $n \to \infty$, $\left(1+\frac{r}{n}\right)^{nt} \to e^{rt}$, as in the compounding interest formula. On the other hand, $1+\frac{r}{n} \to 1$. Hence, the formula for annuities compounded continuously, with a yearly payment $\mathcal{P}$ and a yearly interest rate r, is

$$A(t) = \frac{\mathcal{P}}{r}\left(e^{rt}-1\right)$$

2.2 Loan repayment

Similarly, we have

$$\left(L - \frac{p}{\left(\frac{r}{n}\right)}\right)\left(1+\frac{r}{n}\right)^m + \frac{p}{\left(\frac{r}{n}\right)} = \left(L - \frac{\frac{\mathcal{P}}{n}}{\left(\frac{r}{n}\right)}\right)\left(1+\frac{r}{n}\right)^m + \frac{\frac{\mathcal{P}}{n}}{\left(\frac{r}{n}\right)} =$$

$$= \left(L - \frac{\mathcal{P}}{r}\right)\left(1+\frac{r}{n}\right)^{\frac{n}{r}rt} + \frac{\mathcal{P}}{r} \to \left(L - \frac{\mathcal{P}}{r}\right)e^{rt} + \frac{\mathcal{P}}{r}$$

You may notice that this is, quite naturally, the same as $Le^{rt} - A(t)$. You can check how long it will take to repay the loan, and compare the result with the one in the discrete case.