Differential Calculus for Functions of Two (and more) Variables.

1 Reminder for functions of one variable

As you know, a function of one variable is differentiable at $x = a$ if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ exists and is finite. This is completely equivalent to saying that $f(x) = f(a) + L(x-a) + E_a(x)$, for some finite number L (which turns out to be precisely equal to $f'(a)$), and an "error term" E_a such that $\lim \frac{E_a(x)}{x-a} = 0$, that is, it "goes to zero faster than $x - a$ when $x \to a$". The first definition is convenient for computing $f'(a)$, the second makes the computation useful, since it gives a method for approximating $f(x)$, if we know $f(a)$, and $f'(a)$ (this can be put to good use, for example for estimating $\sqrt{1+x}$ for small x). It also is how we find the tangent to the graph of $f(x)$ at the point $x = a$.

2 The 2-dimensional plane has much more room than the 1-dimensional line

The discussion for functions on one variable is made simple by the fact that you can approach the point a in two ways only – from the left or from the right. Instead, given a point in the plane (a, b) there are infinitely many ways in which a point (x, y) could approach (a, b) along a curve.

This makes the differential calculus in two dimensions much trickier than our familiar one-dimensional approach.

The naive approach is to think of a function of two variables $f(x, y)$, as both a function of x and a function of y and, for example, define it to be "continuous at (a, b)" if $\lim_{x\to a} f(x, b) = f(a, b)$, and $\lim_{y\to b} f(a, y) = f(a, b)$. Similarly, you could define it to be "differentiable at (a, b)" if it was separately differentiable in x and in y – in other words, if it had partial derivatives at (a, b). The book might give you the impression that that's exactly the case (see page 194), but you would be very wrong if you believed that, as we can show with a few examples.

2.1 The proper definition of "continuous" and of "differentiable".

To be formal, the previous "definitions", based on separate behavior of the variables, could well be adopted – definitions are arbitrary, by definition (pardon the pun). However, the point is that the "definitions" above are of practically no use. Let's think about it: the beauty of continuity in one dimension is that if x is sufficiently close to a, $f(x)$ is going to be close to $f(a)$, allowing approximations and related useful facts. In two dimensions, (x, y) is close to (a, b) if both x is close to a and y is close to b *simultaneously*. The simplest way to make sure that's the case is to check that the distance between (x, y) and (a, b) is small, that is check that $\sqrt{(x-a)^2 + (y-b)^2}$ is small[1]. Unfortunately, the "definitions" in the previous section do not enforce this, and it is easy to define a function that is "continuous" in the two variables separately, but takes very different values from $f(a, b)$ for (x, y) arbitrarily close to (a, b). What is worse, a function can easily have partial derivatives but, again, take values very different from $f(a, b)$ even when (x, y) is arbitrarily close to (a, b). In other words, having partial derivatives doesn't even guarantee continuity. Note that, in analogy with the one-dimensional case, we would like a function that is differentiable at (a, b) to have a *tangent plane* there, that is a plane that closely approximates the graph of f for (x, y) close to (a, b), but if the function is not continuous, it will display a jump, a discontinuity in its graph and there is no way that a plane could approximate that. As we will see shortly, having partial derivatives does not guarantee that a tangent plane exists (correcting the impression you might have gotten from the box "Approximating function values with partial derivatives" at page 194).

Continuity

The proper definition of continuity, given the discussion above goes as follows:
a function $f(x, y)$ is continuous at (a, b) if

$$\lim_{(x-a)^2+(y-b)^2 \to 0} f(x, y) = f(a, b)$$

that is, if $f(x, y)$ is close to $f(a, b)$ as soon as the distance between (x, y) and (a, b) is small enough.

[1] There are many other equivalent ways to express closeness, for example require that $|x-a|+|y-b|$ is small (it is easy to see that if this is small, so is the distance, and vice-versa). We'll stick with the distance for simplicity

Differentiability

Planes are described by *linear functions in two variables*, so a plane going through the point (a, b) will be described by a function of the form $A(x-a) + B(y-b)$, where A and B are two numbers. That suggests the proper definition of "differentiable", in analogy with the second definition we mentioned in the one-dimensional case:

a function $f(x, y)$ is differentiable at (a, b) if there exist two numbers A and B such that

$$f(x, y) = f(a, b) + A(x-a) + B(y-b) + E_{(a,b)}(x, y)$$

with

$$\lim_{(x-a)^2+(y-b)^2 \to 0} \frac{E_{(a,b)}(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

It is a simple exercise to verify that if f is differentiable, then $A = f_x(a, b)$, and $B = f_y(a, b)$, so a differentiable function has partial derivatives. Unfortunately, the converse is false, as we will now see.

2.2 Discontinuous functions with partial derivatives

There are a number of examples to show that partial derivatives, by themselves, are basically irrelevant. Even worse, there are easy examples of functions $f(x, y)$ such that, if we approach (a, b) along lines, or even along fancier curves, are continuous (and since they are essentially functions of one variable when evaluated like that, are differentiable in that one variable), but still are discontinuous.

1. A simple example. Define

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

   Since $f(0, y) = f(x, 0) = 0$, the function is continuous in the two variable separately (it is constant on both cases), and has partial derivatives (both equal to 0). However, if we let (x, y) approach $(0, 0)$ along a straight line, say $y = mx$, the function evaluates on this line as

$$\frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1 + m^2}$$

   that is a non zero number when $m \neq 0$, no matter how close to $(0, 0)$ the point (x, mx) is.

2. It gets worse. Define

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

As before, $f(0,y) = f(x,0) = 0$, so we have zero partial derivatives. But now, if we approach $(0,0)$ along a straight line $y = mx$ the function evaluates as

$$f(x,mx) = \frac{mx^3}{x^4 + m^2x^2}$$

whose limit as $x \to 0$ is clearly 0. Moreover, if we compute the derivative at 0 of this function of x (since the definition is piecewise, we have to actually evaluate $\frac{f(x,mx)-f(0,0)}{x}$), we find

$$\lim_{x \to 0} \frac{mx^2}{x^4 + m^2x^2} = \frac{1}{m}$$

(this is called the *directional derivative* along the line $y = mx$). So this function is continuous, and, in fact, differentiable, along any straight line going to the origin. However, if we look at its behavior along the curve $y = x^2$, it evaluates as

$$\frac{x^4}{x^4 + x^4} = \frac{1}{2}$$

no matter how close to 0 x may be!

3. The previous example is just the beginning of a host of similar functions, that are "continuous" along many curves, but are still discontinuous. A strong example relies on the fact that $\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^r} = 0$, for any r. The function defined by

$$f(x,y) = \begin{cases} \frac{e^{-\frac{1}{x^2}} y}{e^{-\frac{2}{x^2}} + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

has, again, partial derivatives equal to 0, but also is "continuous", and, in fact, "differentiable", if evaluated along any curve $y = x^r$, but evaluated along $y = e^{-\frac{1}{x^2}}$, is constant, equal to $\frac{1}{2}$ no matter how close to $(0,0)$ we get.

4. You can work out many other similar examples by choosing a convenient function $h(x)$, such that $\frac{h(x)}{x} \to 0$, as $x \to 0$ (you may make it go to 0 as fast as you wish, to make strong examples, $e^{-\frac{1}{x^2}}$ being a very rapidly vanishing function[2]), and define

$$f(x,y) = \begin{cases} \frac{h(x)y}{h^2(x)+y^2} & (x,y) = (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

2.3 Are partial derivatives useless?

Of course not. But the fact is that for differentiability, *existence* of partial derivatives is not enough. It turns out, though that the following very useful theorem holds:

If the function $f(x,y)$ has partial derivatives at (a,b) ***that are continuous****, the function is differentiable at (a,b).*

Of course, all the examples you see in the book fall in this category.

Note Continuity of the partial derivatives is *sufficient* for differentiability, but is not *necessary.* On can produce fairly elaborate examples of functions that are differentiable at a point, but whose partial derivatives are not continuous there. That said, in areas where we don't require very rigorous hair-splitting precision, like engineering, economics, and so on, the practical definition of "differentiable" is "has continuous partial derivatives", and that's how we are proceeding in this course.

2.4 By the way...

Differentiability is just one example of how the transition from one to two (or more) dimensions is non trivial. Another interesting example consists in the study of critical points. In one dimension, a function with enough derivatives, has a clear behavior at a critical point if it has a non zero derivative of some order there (so $e^{-\frac{1}{x^2}}$ does not work): if the first non zero derivative is of even order (2,4,...), the critical point is a maximum (if the derivative is negative) or a minimum (if it is positive; if the first derivative is of odd order (3,5,...), the point is a horizontal inflection point, increasing if the derivative is positive, decreasing if it is negative). In two, or higher dimensions, we know how to classify a critical point if $D = f_{xx}f_{yy} - f_{xy}f_{yx} \neq 0$ (assuming all derivatives are continuous as this is now our standing assumption). When $D = 0$, it turns out that there are many possibilities, and none is recognizable with as simple a rule as that for one dimensional critical points. In fact, at some point in the 70s, this was a popular topic of research, under the catchy title of "Catastrophe Theory" (the motivation of this peculiar name is available on request).

[2] You can read about $e^{-\frac{1}{x^2}}$ in another item, "zeroderivative.pdf", in this "Additional Materials" collection, as a popular example. Precisely, if we define $k(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$, k is obviously continuous, but it also has all derivatives, all infinitely many of them, and all such that $k^{(m)}(0) = 0$, $m = 1, 2, \ldots$, even though it is not zero for any $x \neq 0$.