Determining a Triangle

1 Constraints

What data do we need to determine a triangle? There are two basic facts that constrain the data:

1. The triangle inequality: *The sum of the length of two sides is greater than the length of the third side.* A consequence is also that *The difference of the lengths of two sides is greater than the length of the third side.* In formulas,
$$a + b > c$$
$$|a - b| < c$$

2. *The sum of the three angles in a triangle is equal to* $180°$*, that is, in radians,* π

3. *The side opposite to the largest angle is the longest, and the one opposite to the smallest is the shortest.*

With these constrains in mind, it turns out that if we are given three independent consistent data (sides and/or angle sizes), the other three are determined (with a small exception). This can be seen geometrically, basically using "ruler and compass constructions", or analytically, using trigonometry.

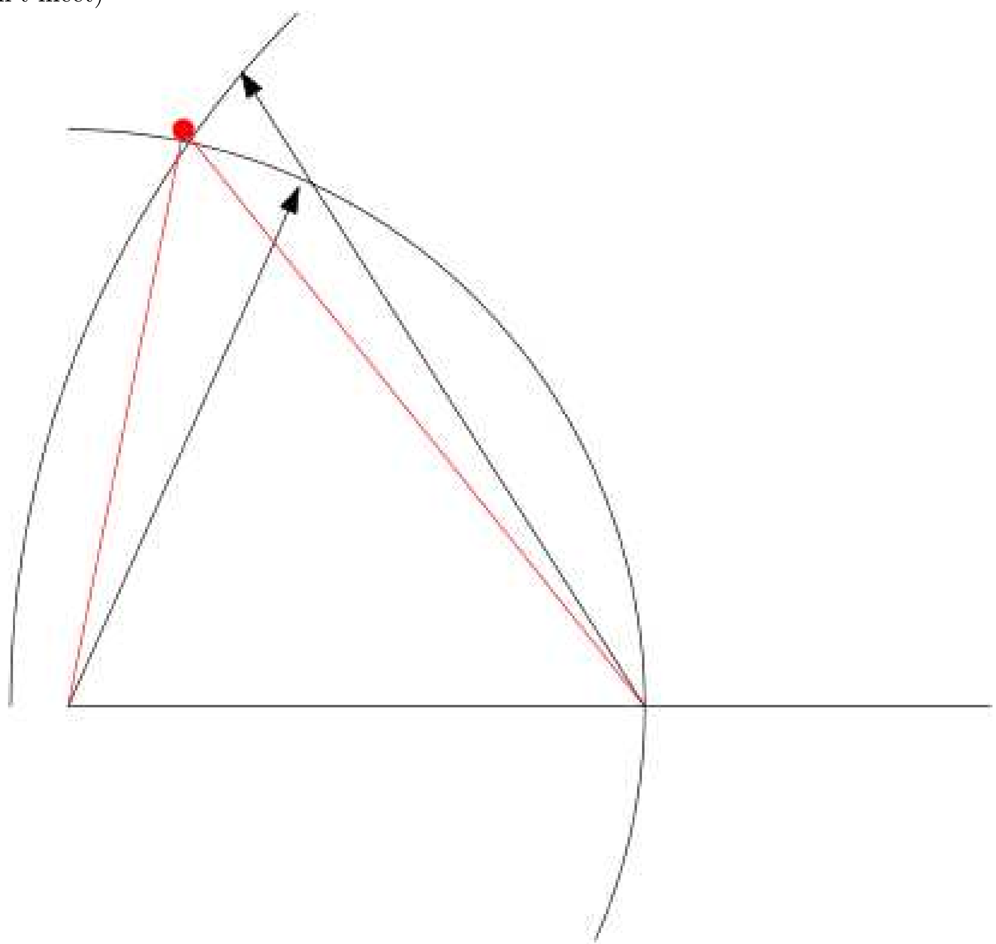
2 Determining a Triangle

First note that "three independent data", rules out that three angles would suffice: they are not independent, because of point 2 above (we are actually only assigning **two** as the third is then fixed). Assigning only angles determines only the "shape" of the triangle, since two triangles with equal angles are *similar*.

Call the three sides a, b, c and the angles opposite to them, respectively, α, β, γ.

2.1 Three sides

Assigning three sides (subject to the triangle inequality) determines the triangle. The following picture suggests how you would draw the triangle, given three sides, using a compass. If the triangle inequality is not satisfied (side c is either too long ($c > a + b$) or too short ($c < |a - b|$)), the two circles in the picture don't meet)



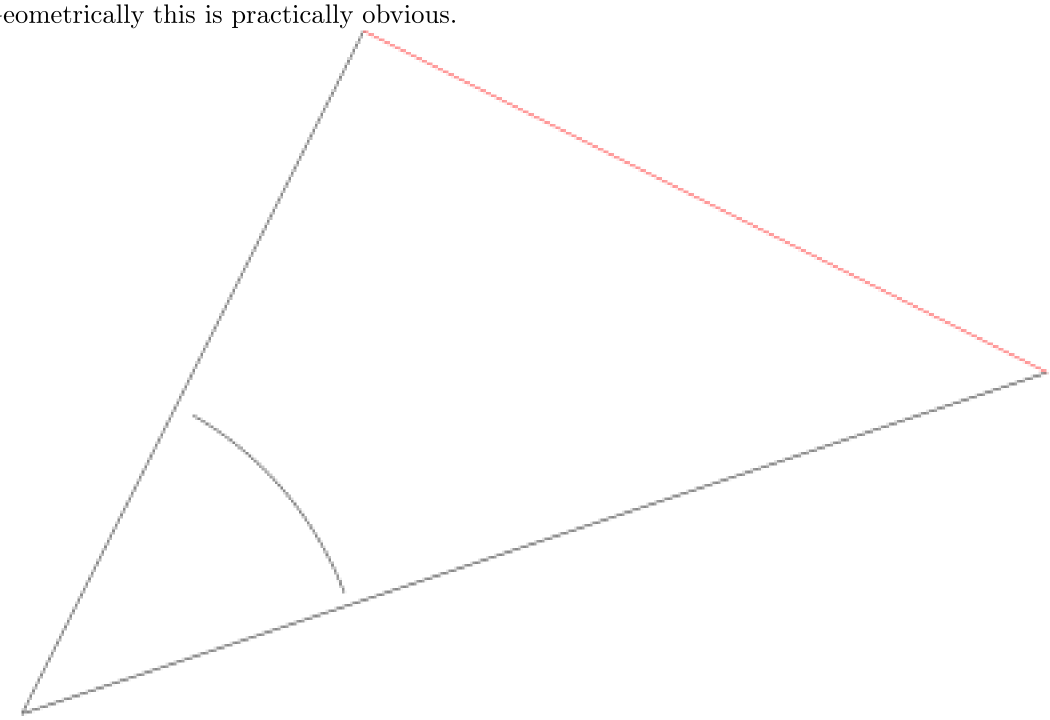
Analytically, we use the law of cosines, since

$$\begin{cases} \cos\alpha &= \frac{b^2+c^2-a^2}{2bc} \\ \cos\beta &= \frac{a^2+c^2-b^2}{2ac} \\ \cos\gamma &= \frac{a^2+b^2-c^2}{2ab} \end{cases}$$

Note that *Heron's Formula* allows us to compute the are of the triangle right away with this data.

2.2 Two sides and the angle between

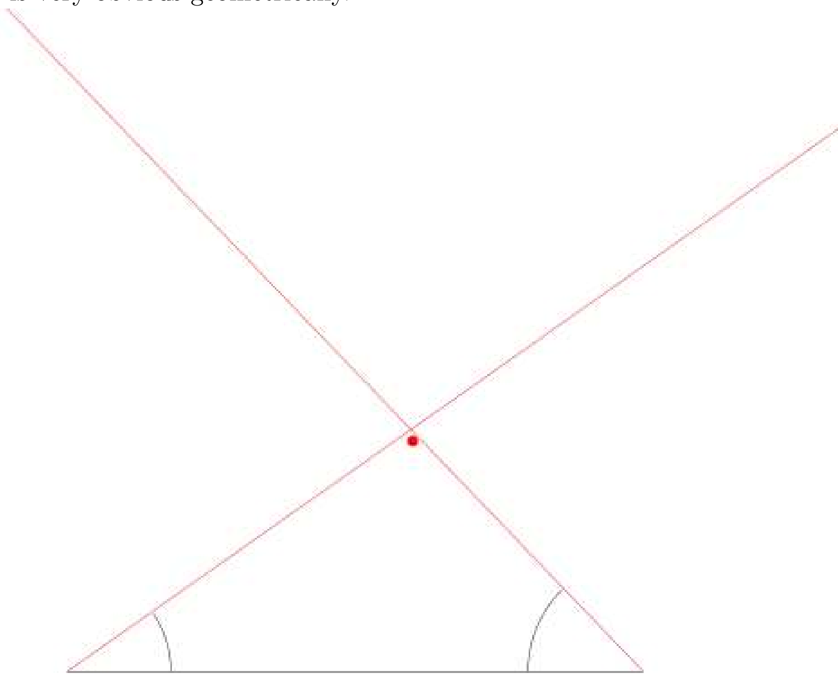
Geometrically this is practically obvious.



Analytically, we can again apply the law of cosines to determine the third side, and proceed from there following point 2.1, or apply the law of sines to determine the remaining angles: for example, knowing a, b, c and γ, $\sin\alpha = \frac{a}{c}\sin\gamma$. Note that for this construction the only constraint is that the angle be less than π, or 180°.

2.3 One side and the two adjacent angles

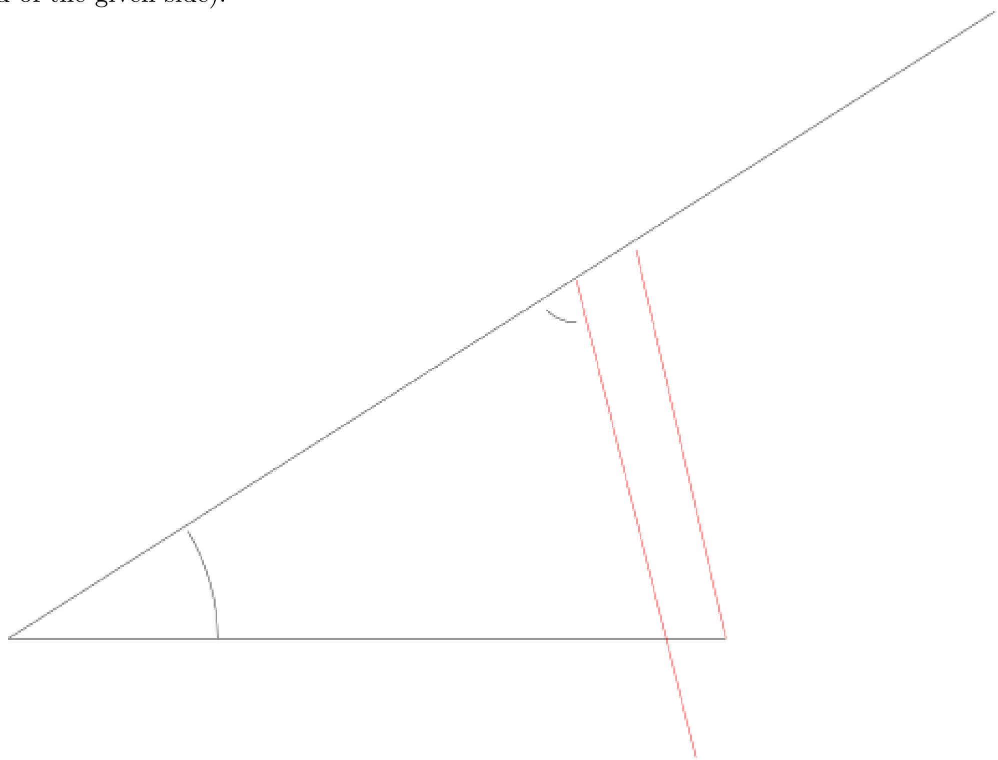
This two is very obvious geometrically.



Analytically, the law of sines yields the remaining two sides right away, while the third angle is given simply as the complement to 180° to the sum of the two. Here, the constraint is that the *sum* of the two angles be less than π or 180°

2.4 One side, an adjacent, and the opposite angle

The geometric construction is only slightly more elaborate (slide the opposite angle along the side of the adjacent one, until the remaining side touches the end of the given side).



Analytically, it's again the law of sines that helps us: given a, β and α, $\gamma = 180° - (\alpha + \beta)$,while $b = \frac{a \sin \beta}{\sin \alpha}$, and $c = \frac{a \sin \gamma}{\sin \alpha}$. Of course, this is actually the same situation as in case 2.3, since assigning two angles assigns the third as well, so γ is also known, and the constraint is also the same.

2.5 Two sides and an angle adjacent to one side, and opposite to the other

Suppose a, b, α are the given data. First note that if $\alpha \geq 90°$, this is the greatest angle, so a must be the longest side, and there is "no room" for the other angles to take more than one value (they are both acute). Thus, in this case, a triangle is uniquely determined if and only if $b < a$, otherwise no triangle satisfies the data. The interesting case is if α is acute.This is a data set that is not always enough to identify the triangle. Depending on the data, no triangle is determined (if the data is inconsistent), one triangle is determined, or there are two possible triangles! The following pictures show the three possibilities.

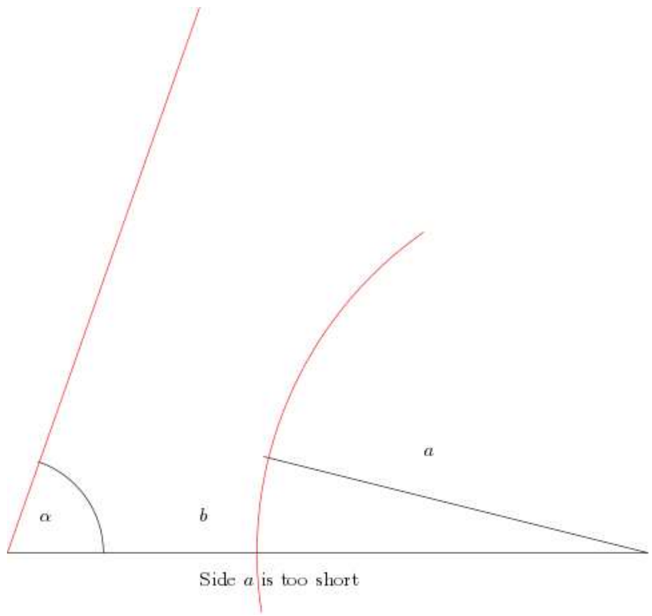


Fig. 1: Inconsistent data

It is clear that this situation lasts as long as a is so short that it cannot touch the side of α. This situation ends when a just reaches to the side, in which case it will be perpendicular the side and hence will have length $a = b \sin \alpha$. Thus, there is no triangle if $a < b \sin \alpha$.

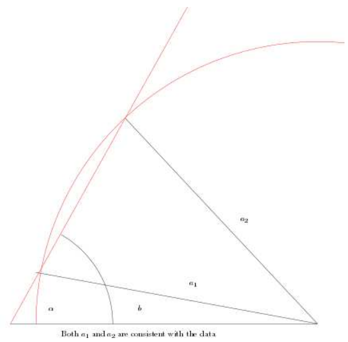


Fig. 2: Two triangles are possible

If $a > b \sin \alpha$, we can have two possible positioning of side a, labeled a_1 and a_2 in the picture, at least for a certain range of lengths for a. The limit case occurs when the circle described by the side a ends up at the vertex of the angle α, that is when $a = b$. Thus there are two possible triangles as long as $b \sin \alpha < a < b$.

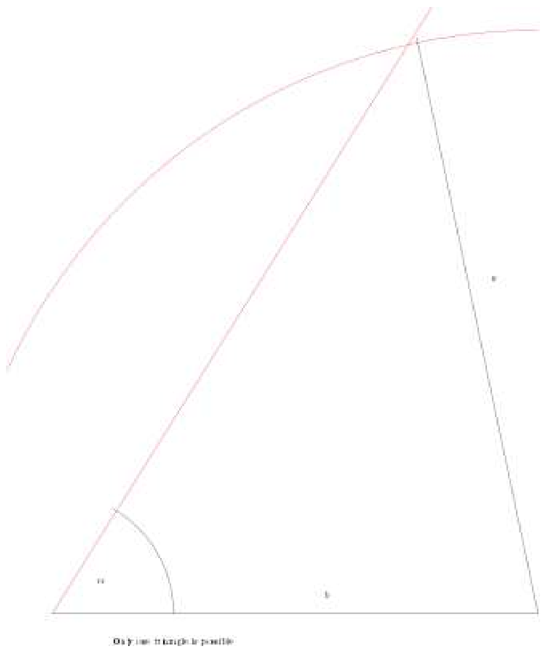


Fig. 3: Only one triangle is possible

As discussed, this situation occurs if $a \geq b$.

Analytically, suppose again the data are the sides a and b and the angle α (as in the figures). The law of sines implies that the angle β, opposite to b, and adjacent to a, has to be such that

$$\sin\beta = \frac{b\sin\alpha}{a} \tag{1}$$

If $a < b\sin\alpha$, the right hand side is greater than 1, and no angle β will satisfy that.

If $a = b\sin\alpha$, $\sin\beta = 1$, and the only solution is $\beta = 90°$.

If $a > b\sin\alpha$, there are two angles satisfying (1), $\beta_1 = \sin^{-1}\left(\frac{b\sin\alpha}{a}\right)$, an acute angle, and $\beta_2 = 180° - \beta_1$ an obtuse angle. The third angle γ then, is $\gamma_1 = 180° - (\alpha + \beta_1)$ or, respectively, $\gamma_2 = 180° - (\alpha + 180° - \beta_1) = \beta_1 - \alpha$. For γ_2 to be positive, hence a proper angle for a triangle, we need $\beta_1 > \alpha$, in which case we have two triangles, with the two sets of angles listed (the third side follows from the law of sines or the law of cosines), which in turn requires $b > a$. If $a \le b$, there is no triangle with angle γ_2, and we have a unique solution to our problem.