What functions f are such that $f(x+y) = f(x) + f(y)$?

1 Introduction

A very common, but very bad, mistake is to write that, for example, $(x+y)^2$ should be equal to x^2+y^2. It is easy to see that that's very wrong: just set $x=y=1$, and notice that $(1+1)^2 = 2^2 = 4$, while $1^2+1^2 = 1+1 = 2$. Similarly, we see all too often work where $\sqrt{x+y}$ is, wrongly, set equal to $\sqrt{x}+\sqrt{y}$. Again, just setting $x=y=1$ shows this is very wrong ($\sqrt{1+1} = \sqrt{2}$, while $\sqrt{1}+\sqrt{1} = 1+1 = 2$). There are many more examples of this type. The fact is that only a very restrictive class of functions satisfies the (lazy :) equality $f(x+y) = f(x)+f(y)$: with an "exotic" exception, the only function satisfying this equality are *linear functions*, of the form $f(x) = Kx$.

A complete answer to the question, using not too exotic tools (but, still, not really light fare) can be found in the paper: E.Hewitt and H.S.Zuckerman *Remarks in the Functional Equation $f(x+y) = f(x)+f(y)$. Mathematics Magazine* **42** No.3 (May, 1969), 121 – 123.

The full story is in this theorem: THEOREM. *If f is a solution of (1) not of the form $f(x) = Kx$, then the graph of $y = f(x)$ has a point in every neighborhood of every point in the plane $\mathbb{R}\times\mathbb{R}$.*

In other words, the only functions not of the form $f(x) = Kx$ (linear functions) satisfying that equality are very pathological, and such that you are not likely to come across very often. As a matter of fact, the theorem, as stated, does not even imply that such strange functions actually exist. They have been constructed by K.G.Hamel (in 1905) using a much more advanced (and not entirely non controversial[1]) toolbox.

While the result above is essentially complete, but requires some delicate arguments, we can easily show that any "decent" function such that

$$f(x+y) = f(x)+f(y) \tag{1}$$

holds has to be linear, that is of the form

$$f(x) = Kx \tag{2}$$

for some real number K (note that $K=0$ is perfectly acceptable).

[1] Actually, the vast majority of mathematicians are perfectly at ease with proofs using the *Axiom of Choice*, however if a proof is available without referring to it, it is definitely preferred.

2 If f is nice..

First let us note that if $f(x) = Kx$, for some real number K, then

- $f(0) = 0$
- $f(1) = K$

Next, if (1) holds, then, for any x,

$$f(x+0) = f(x) = f(x) + f(0)$$

which implies

$$f(0) = 0$$

A quick proof when f is differentiable

This is direct: for any x

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(f(x+h) - f(x)\right) = \lim_{h\to 0} \frac{1}{h}\left(f(x) + f(h) - f(x)\right) = \lim_{h\to 0} \frac{f(h)}{h} = f'(0)$$

(since $f(0) = 0$). Hence, the derivative function is constant, which means that $f(x) = Kx + b$. But $f(0) = 0$, so $b = 0$. By the way, looking at the proof, we can notice that, given (1), if f is differentiable at one point, it is differentiable everywhere (also, see below for the similar statement concerning continuity).

A curious fact: (1) implies (2) for any rational numbers x and y

This fact points out that the possible odd behavior of our functions is concentrated on the irrational numbers (but, in a precise sense, irrational numbers are much more "numerous" than rational ones, so that is a big deal), pointing out that the passage from rationals to reals is less trivial than one would at first imagine.

So, let $x = \frac{m}{n}$ be restricted to be a rational number. Then

$$f(x) = f\left(\frac{m}{n}\right) = f\left(m \cdot \frac{1}{n}\right) = f\left(\sum_1^m \frac{1}{n}\right) = \sum_1^m f\left(\frac{1}{n}\right) = mf\left(\frac{1}{n}\right) =$$

$$= \frac{m}{n} \cdot nf\left(\frac{1}{n}\right) = \frac{m}{n} f\left(\frac{n}{n}\right) = xf\left(1\right)$$

Call $K = f(1)$, and we have proved that $f(x) = Kx$.

Moving beyond rational numbers

For a function *not* to behave like the strange functions mentioned in the first theorem, it needs very little regularity (it turns out, for example, that being Lebesgue measurable is more than enough – this being an advanced property to be investigated in advanced calculus), but we can still prove that a weaker condition than differentiability is enough: continuity at one point.

Assume f is continuous at one point a

Consequence f is continuous everywhere.

In fact, we have that $f(a+h) \to f(a)$ as $h \to 0$. But $f(a+h) = f(a) + f(h)$, so $f(h) \to 0$. Hence, for any x, $f(x+h) = f(x) + f(h) \to f(x)$ as $h \to 0$.

If our f is continuous, for any real number x, we can find a sequence of rational numbers $r_n \to x$. By continuity, $f(x) = \lim f(r_n) = \lim K r_n = Kx$.

3 Another condition for linearity

Linearity also implies that

$$f(ax) = af(x) \tag{3}$$

(*homogeneity of degree* 1*)* for any a and x. In fact, you will often see the definition of a linear function to be one that satisfies $f(ax+by) = af(x)+bf(y)$ for any a, b, x, y. Fact is that homogeneity implies linearity: if (3) holds for any a and x, then

$$f(x) = f(x \cdot 1) = xf(1)$$

and calling $K = f(1)$, $f(x) = Kx$.

4 A related issue: what functions are such that $f(x+y) = f(x)f(y)$

The property

$$f(x+y) = f(x)f(y) \tag{4}$$

is typical of exponential functions. It turns out that, with a similar exception as for the case discussed above, the only "reasonable" functions satisfying (4) are exponential functions. Recall that you can represent any exponential function using one base, and we will choose e, as it is the *natural* base.

To go through the argument, we first notice that one possibility for f is that $f(x) = 0$ for all x. On the other hand, suppose $f(a) = 0$ for some a, then for any other x,

$$f(x) = f(x-a+a) = f(x-a)f(a) = 0$$

so $f(x) = 0$ for all x. Excluding this case, $f(x) \neq 0$ for all x. But we also have

$$f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = f\left(\frac{x}{2}\right)^2 > 0 \tag{5}$$

so $f(x) > 0$ for all x.

It has to be an exponential

We could mimic all the arguments in section 2, but it is much faster to apply the following arguments: if f satisfies (4), taking logarithms (we'll choose natural logs, but any would do), which we can, thanks to (5), we have

$$\ln f\,(x+y)=\ln f\,(x)+\ln f\,(y)$$

so, by section 2, except the case of a very irregular f, and consequently $\ln f$,

$$\ln f(x)=Kx$$

for some real number K. That means that

$$f(x)=e^{Kx}$$