A Quick Note on Taylor's Formula

Some of you may not be very familiar with Taylor's Formula, and especially its implications. Here is a very quick statement of the formula, with an example of how you can apply it to shorten some calculations.

Theorem Suppose $f(x)$ is a function with m continuous derivatives at x_0, a point in the interior of its domain. Then there is one and only polynomial $T_m(x)$such that

$$\frac{|f(x) - f(x_0) - T_m(x)|}{(x - x_0)^m} \to 0$$

when $x \to x_0$. The k-th term of the polynomial ($k \leq m$) is $\frac{f^{(k)}}{k!}(x - x_0)^k$.

Now, this might not seem much, but what it says is that when x is very close to x_0, $f(x)$ should look very much like $f(x_0) + T_m(x)$. Here are two well-known examples:

1. Take $m = 1$. $T_1(x) = f'(x_0)(x - x_0)$. This is a linear function, increasing if $f'(x_0) > 0$, decreasing if $f'(x_0) < 0$. In fact, f will behave exactly in the same way near x_0. Actually, more than that, $f(x_0) + T_1(x_0)$ is the tangent line at x_0

2. Take $m = 2$. $T_2(x) = f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$. This is a parabola, "open up" or "open down", depending on whether $f''(x_0)$ is positive or negative. If $f'(x_0) = 0$, the parabola $f(x_0) + T_2(x)$ has its vertex at x_0, and you will recognize the familiar rule to determine maxima and minima. If $f'(x_0) > 0$, the parabola is increasing, and hence it is concave up, while if $f'(x_0) < 0$ it will be concave down. Both facts apply to the function f, as is well known.

3. From a general point of view, Taylor's Theorem means that the successive derivatives of a function *are the coefficients of the polynomial that "best" approximates it* (in the sense of the theorem). The point is that polynomials are relatively easy to study and describe, and the core idea of differential calculus is to substitute an "easily manageable" polynomial to a function which, in general, would be much more difficult to describe. The more derivatives a function has, the more subtle the description we will be able to give. If *all* derivatives exist (such functions are called C^∞), the we can push this tool to any level we desire. This approach becomes extremely useful when studying functions of several variables (using the multi-dimensional Taylor Theorem).

4. Note that C^∞ is entirely different from "real analytic" (sometimes written as C^ω), that is functions that are *the sum of a convergent Taylor series.* Real analytic functions are, obviously, C^∞, but the converse is far from true. While all elementary functions (meaning exponentials, logarithms, trigonometric functions) are real analytic, there are plenty of counterexamples. The most famous (you will definitely meet it) is the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   It has derivatives of any order at $x = 0$, and they are all equal to zero. Hence, its Taylor expansion is trivial, but the function is obviously not zero except at the origin.

There is more in this direction (at least in the case of functions of one variable[1] – the case of functions in several variables is notably more complex, but also extremely interesting), but we'll finish on a simple application of all this to L'Hospital's formula (which is a bit more general, but in most cases this "Taylor" version is good enough).

Suppose $f(x_0) = g(x_0) = 0$, and both functions have one continuous derivative. Then

$$\frac{f(x)}{g(x)} = \frac{f(x_0) + f'(x_0)(x - x_0) + R(x, x_0)}{g(x_0) + g'(x_0)(x - x_0) + S(x, x_0)}$$

with both R and S vanishing at a faster rate than $(x - x_0)$, as $x \to x_0$. Dividing numerator and denominator by $x - x_0$, we easily see that

$$\frac{f(x)}{g(x)} = \frac{f'(x_0) + \frac{R(x,x_0)}{x-x_0}}{g'(x_0) + \frac{S(x,x_0)}{x-x_0}} \to \frac{f'(x_0)}{g'(x_0)}$$

It is easy to extend this argument to the case when the first few derivatives of the two functions are zero.

In summary, Taylor's theorem lets us use the properties of approximating polynomials to study the local behavior of functions that are much more complicated. To this end, it is useful to remember some of the simplest examples:

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$
- $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$
- $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \ldots$

[1] For example, a remarkable fact, true only in one dimension, is that if $f^{(k)}(x_0) = 0$ for $k = 1, 2, \ldots n - 1$, and $f^{(n)}(x_0) \neq 0$, *there is a nonlinear change of variables in a neighborhood of* x_0, $y = \varphi(x)$, *such that, in the new coordinates,* $f(y) - f(y_0) = \frac{1}{n!} f^{(n)}(y_0)(y - y_0)^n$ *exactly.* The proof relies on the Implicit Function Theorem, which is a deep result in multivariate calculus, but, once the theorem is a given, is not hard.

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

For example, you can easily see that, as $x \to 0$

- $\frac{e^x - 1}{x} \to 1$ (that is, $e^x \approx 1 + x$)
- $\frac{\log(1+x)}{x} \to 1$
- $\frac{\sin x}{x} \to 1$ (we knew that)
- $\frac{1-\cos x}{x} \to 0$, $\frac{1-\cos x}{x^2} \to \frac{1}{2}$

and more interesting limits (all the above are essentially the computation of derivatives).