A Very Superficial Observation

The CLT is a good argument for suggesting that the normal distribution is, in some vague sense, a first (non constant) approximation to any well-behaved distribution (this is actually an idea that can be pursued in a specific sense: one can put a differentiable structure on a class of "well-behaved" distributions, and Gaussian distributions turn out to form the "tangent space" to this structure).

A totally hand-waving way to reinforce this notion is the following.

Let's restrict to distributions with a nice density f that is positive everywhere (hence, for any interval I, $P[\mathbb{R} \setminus I] < 1$, strictly). It follows that $\phi = \log f$ is well-defined everywhere, and we can write

$$f(x) = e^{\phi(x)} \tag{1}$$

Now, if f (and hence ϕ) has two continuous derivatives, we have that

$$\phi(x) = \phi(0) + \phi'(0)x + \frac{\phi''(0)}{2}x^2 + o\left(x^2\right) = a + bx + cx^2 + \varepsilon$$

If we decide to ignore ε, our density is approximated by

$$f(x) \approx e^{a+bx+cx^2}$$

By completing the square, this is a Gaussian distribution (of course, necessarily $c < 0$, or else this would not be integrable).

Why would we be willing to neglect a term $o\left(x^2\right)$? For example, we would do that in the context of the rescaling implicit in the CLT.

Remark: Why didn't we stop at the *first* order? Well, if we assume that $f(x) > 0$ for *all* x, this won't work, as the resulting "approximation" cannot be a density.

Remark: Actually, the family of distributions such that (1) makes sense is very popular in Mathematical Statistics: it is called the *Exponential Family.*

Disclaimer: This is not only not a proof of anything, and it is not even a real argument for the statement that "normal distributions are 'tangent' to 'nice' distributions". However, it looks like a fun observation...