Independent Events

Two events A and B are said to be ***independent*** if the probability of occurrence of one event is not affected by the occurrence of the other event, that is,

$$P\{A|B\} = P\{A\} \qquad \text{and} \qquad P\{B|A\} = P\{B\} \qquad (*)$$

where $P\{A\}$ and $P\{B\}$ are assumed nonzero. An equivalent but more compact form of $(*)$ is

$$P\{A \cap B\} = P\{A\}P\{B\} \qquad (**)$$

Thus, formally two events A and B are said to be statistically (or probabilistically) ***independent*** if $(**)$ holds. The equivalence of $(*)$ (with $P\{A\} \neq 0 \neq P\{B\}$) and $(**)$ can be seen as follows:

$$(2.17) \implies P\{A \cap B\} \overset{(2.14)}{=} P\{A|B\}P\{B\} \overset{(2.17)}{=} P\{A\}P\{B\} \implies (**)$$

$$(2.18) \implies \left\{ \begin{array}{l} P\{A\}P\{B\} \overset{(2.18)}{=} P\{A \cap B\} \overset{(2.14)}{=} P\{A|B\}P\{B\} \\ P\{A\}P\{B\} \overset{(2.18)}{=} P\{B \cap A\} \overset{(2.14)}{=} P\{B|A\}P\{A\} \end{array} \right\} \implies (*)$$

Events are said to be statistically ***dependent*** if they are not independent. Independence simplifies the calculation of joint probability greatly:

$$\text{joint probability} \overset{\text{if independent}}{=} \text{product of probabilities}$$

Independence of n Events

For n events $A_1, A_2, \ldots, A_n$, if

$$\begin{aligned} P\{A_i \cap A_j\} &= P\{A_i\}P\{A_j\} \qquad \forall i \neq j \\ P\{A_i \cap A_j \cap A_k\} &= P\{A_i\}P\{A_j\}P\{A_k\} \qquad \forall i \neq j \neq k \\ &\vdots \\ P\{A_1 \cap A_2 \cap \cdots \cap A_n\} &= P\{A_1\}P\{A_2\} \cdots P\{A_n\} \end{aligned}$$

then events $A_1, A_2, \ldots, A_n$ are said to be statistically ***independent***. Otherwise, they are dependent.

Independent vs. Disjoint Events

Can two events be both independent and ***disjoint*** (i.e., ***mutually exclusive***)? Note that

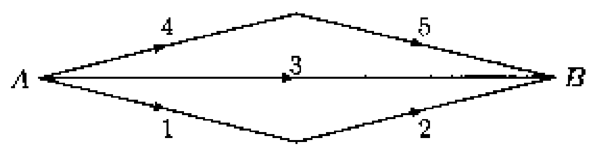
$$P\{A\}P\{B\} \overset{\text{independent}}{=} P\{A \cap B\} \overset{\text{disjoint}}{=} 0$$

indicates that if two events are both independent and disjoint, then at least one of them has zero probability — *nonzero-probability events cannot be both independent and disjoint*. Intuitively, if two events are disjoint, the occurrence of one precludes the other and thus they cannot be independent. Note the difference:

$$\begin{aligned} \text{joint probability} &\overset{\text{if independent}}{=} \text{product of probabilities} \\ \text{union probability} &\overset{\text{if disjoint}}{=} \text{sum of probabilities} \end{aligned}$$

Example 2.17: Reliability of Communication Channel

Consider the following communication network. Assume the links are independent and the probability that a link is operational is 0.95.



Since independence of links implies that paths are independent, the probability of being able to transmit from A to B can be calculated as follows:

$$\begin{aligned} P\{\text{path 1-2 OK}\} &\overset{?}{=} P\{\text{link 1 OK}\}P\{\text{link 2 OK}\} = 0.95 \times 0.95 = 0.9025 \\ P\{\text{path 1-2 fails}\} &= 1 - P\{\text{path 1-2 OK}\} = 1 - 0.9025 = 0.0975 \\ P\{\text{path 3 fails}\} &= 1 - P\{\text{link 3 OK}\} = 1 - 0.95 = 0.05 \\ P\{\text{all paths fail}\} &\overset{?}{=} P\{\text{path 1-2 fails}\}P\{\text{path 4-5 fails}\}P\{\text{path 3 fails}\} \\ &= 0.0975 \times 0.0975 \times 0.05 = 0.000475 \end{aligned}$$

Finally,

$$\begin{aligned} P\{\text{able to transmit from } A \text{ to } B\} &= 1 - P\{\text{all paths fail}\} \\ &= 1 - 0.000475 = 0.999525 \quad \text{(very high)} \end{aligned}$$