Difference Equations and Random Walks

Math 394

We look at an unexpected connection between random walks and difference equations. If we take this connection to the limit of infinitesimally small steps, we end up looking at second order differential equations. While all this seems a coincidence, it turns out to hint at a deep connection between a certain important class of "stochastic processes" (that is, random objects, changing over time) and differential equations – yet another connection between Probability and Analysis, this time less obvious than the "obvious one", between axiomatic Probability and Measure Theory.

1 Reminder on Random Walks

In a separate file, we discussed the problem of "Gambler's Ruin", which is also known as the "Random Walk". There are many questions that can be asked (and most can be answered) about this model. One tool that was used to derive results was the following (the meaning of the symbols is discussed in the original file):

The function $\pi(x)$ (the probability of ruin of Player #1 if x is her current wealth), under the conditions detailed in the file, satisfies the following "boundary value problem":

$$p \cdot (\pi(x+1) - \pi(x)) = (1-p)\,(\pi(x) - \pi(x-1)) \tag{1}$$

$$\pi(0) = 1 \qquad \pi(m) = 0$$

2 The Symmetric Case

Suppose $p = \frac{1}{2}$. Equation (1) becomes

$$\frac{\pi(x+1) + \pi(x-1) - 2\pi(x)}{2} = 0 \tag{2}$$

This is a "difference equation" (whose solution is a linear function), whose left hand side is reminiscent of the "second difference" of a function – an expression that has as limit, under appropriate assumptions, the second derivative of the function π.

In fact, in this model the domain of π are the non negative integers. Now, assume we change our scale, so that each step is of size h, instead of 1. The domain of π consists now of integer multiples of $\frac{1}{h}$. Assume we may extend π to be defined over all real numbers in the interval $[0, m]$, as a C^2 function (a continuous function, with two continuous derivatives).

We can also multiply the equation by $\frac{1}{h^2}$, without changing anything, since the right hand side is zero. We get

$$\frac{\pi(x+h)+\pi(x-h)-2\pi(x)}{2h^2}=0 \tag{3}$$

If we assume that π is a function with two continuous derivatives (incidentally, the solution to (2) is linear in x, so it can be extended smoothly to a linear function, that surely satisfies our condition), by Taylor's Theorem,

$$\pi(x+h)-\pi(x)+\pi(x-h)-\pi(x)=\pi'(x)\cdot h+\frac{1}{2}\pi''(x)\cdot h^2-\pi'(x)\cdot h+\frac{1}{2}\pi''(x)\cdot h^2+o\left(h^2\right)$$

(where we use the familiar notation $o(x)$ for some function such that $\lim_{x\to 0}\frac{o(x)}{x}=0$). Substituting into (3), we get

$$\frac{1}{2}\pi''(x)+\frac{o\left(h^2\right)}{h^2}=0$$

If we let $h\to 0$, this equation turns into a 1-dimensional *Laplace equation*, with boundary conditions as above. Clearly, the only classical solutions will be linear functions, and the boundary conditions identify this function uniquely.

3 The Asymmetric Case ("Random Walk With Drift")

The file we are referring to, discusses the solution to (1) in the general case, when $p\neq\frac{1}{2}$. We may wonder what becomes of the discussion presented in section 2

To this end, let us write $p=\frac{1}{2}+\kappa$ (it doesn't matter what the sign of κ may be, of course). Then, $1-p=\frac{1}{2}-\kappa$, and the equation can be written as

$$\frac{\pi(x+1)+\pi(x-1)-2\pi(x)}{2}+\kappa\left(\left[\pi(x+1)-\pi(x)\right]+\left[\pi\left(x\right)-\pi(x-1)\right]\right)=0$$

Now, we have an additional term, involving "first differences".

To repeat the argument in section 2, we need to cheat a little, since, simply changing our steps to size h, will lead to

$$\frac{1}{2}\pi''(x)\cdot h^2+2\kappa\pi'(x)\cdot h+o(h)=0$$

Now, dividing by h, and letting $h\to 0$, leads to a first order differential equation whose solutions are constant, and hence cannot satisfy the boundary conditions. Dividing by h^2 leads to no limit at all.

The "cheating" idea is to adjust the "bias" in the game (the asymmetry between p and $1-p$), as we change h. Precisely, we may set $\kappa = \frac{b}{2} \cdot h$ (as $h \to 0$, the asymmetry of our random walks becomes smaller and smaller). Now, dividing by h^2, and letting $h \to 0$ leads to the equation

$$\frac{1}{2}\pi''(x) + b \cdot \pi'(x) = 0 \tag{4}$$

If you are familiar with linear differential equations with constant coefficients, you will have no trouble finding the general solution to this one: with the characteristic roots being

$$\lambda_1 = 0, \lambda_2 = -2b$$

the general solution is of the form

$$C_1 + C_2 e^{-2bx}$$

and the boundary conditions result in

$$C_1 + C_2 = 1$$

$$C_1 = -C_2 e^{-2bm}$$

$$C_2 = \frac{1}{1 - e^{-2bm}}, C_1 = 1 - \frac{1}{1 - e^{-2bm}} = -\frac{e^{-2bm}}{1 - e^{-2bm}}$$

$$\pi(x) = \frac{e^{-2bx} - e^{-2bm}}{1 - e^{-2bm}}$$

(You can easily check that this is indeed a solution to (4), satisfying the boundary conditions $\pi(0) = 1, \pi(m) = 0$, and that this solution is unique). Hence, the limiting equation has an exponential type solution (not surprisingly, of course, if we recall what the solution to the difference equation (1) turned out to be).