Math 394

A collection of problems about probability and conditional probability

1 Quizzes

1.1

Suppose that $A =$"Seattle will win the division", $B =$"Seattle will win the series with Oakland". The set $(A \cap B)^c$ (the complement of the intersection) is

○ Seattle will not win the division, but it will win the series with Oakland

○ Seattle will win the division and the series with Oakland

⊗ Seattle will not win the division or will lose the series with Oakland or both.

○ Seattle will win either the division, or the series with Oakland, but not both

○ Seattle will win the division, or win the series with Oakland, or possibly both.

Discussion: In general, $(A \cap B)^c = A^c \cup B^c$, and $(A \cup B)^c = A^c \cap B^c$. In our case, we have the *union* ("OR") of the complement of Seattle winning the division and the complement of Seattle winning the series with Oakland. OR means, "either, or, or both", hence the answer.

1.2

Let $P[A] = 0.409$, and $P[B] = 0.612$. Which of the following statements will *always be false*

○ $P[A \cup B] \leq P[A] + P[B]$

○ A and B are independent

⊗ A and B are disjoint

○ $P[B|A] \leq P[B]$

Discussion: The first option is actually always *true*! The second and fourth need a lot more information to be stated, but the one thing we know is that the third is impossible, because, otherwise, $P[A \cup B] = P[A] + P[B] = 1.21 > 1$!

1.3

Let A ="it is raining today", B ="the sky is cloudy today", C ="the sky will be clear tomorrow". Which of the following relations will certainly be true (at least in most circumstances)?

○ $A \cap B = \emptyset$

○ $B \cap C = \emptyset$

○ $C \cap A = \emptyset$

⊗ $A \subseteq B$

○ $B \subseteq A$

○ $A \subseteq C$

○ $C \subseteq A$

○ $B \subseteq C$

○ $C \subseteq B$

Discussion: Like most "real word" logic statements, you could pick on this one. But let us agree that, around here, at least, we can safely say that in case of rain, the sky will be cloudy. As we all know, A, and B being statements about today's weather, they bear little relation to C (sometimes one is tempted to say they are actually independent...), so we can rule out all of the above that mix C with either A or B. Note that saying $A \subseteq B$, means that every ω in A is also in B, so that every time we observe A we necessarily will also observe B. Since it's not true that every time it is cloudy it will also be raining, we have our answer.

1.4

Let A and B be two events. We say they are independent if (choose the answer that is *always* true in this circumstance)

○ $P[A \bigcap B] = P[A]$

○ $P[A \bigcap B] = 0$

⊗ $P[A \bigcap B] = P[A]\, P[B]$

○ $P[A|B] = P[B]$

Discussion: This is just the definition of independence.

1.5

Suppose A and B are two *disjoint* events. Which of the following statements is *always* true:

○ $P(A \cap B) = P(A)P(B)$

○ $P(A|B) = P(A)$

⊗ $P(A \cup B) = P(A) + P(B)$

○ $P(A) = 1 - P(B)$

Discussion: The first two statements are actually *false* (at least for events of positive probability). The fourth would be true only if, besides being disjoint, A and B formed a partition (i.e., $A \cup B = \Omega$). The third (correct) statement is actually one of the possible formulations of the basic axiom of probability theory.

1.6

We toss a fair die repeatedly, until, at the Tth toss we get, for the first time, a **6**. $P(T > 5)$ is equal to:

○ $\left(\frac{1}{6}\right)^5$

⊗ $\left(\frac{5}{6}\right)^5$

○ $\frac{1}{6}\left(\frac{5}{6}\right)^4$

○ $\frac{5}{6}\left(\frac{1}{6}\right)^4$

Discussion: $\{T > 5\}$ means that we did not get a **6** over the first 5 throws. A **6** does not come up with probability $\frac{5}{6}$ at each toss, and we normally assume that such tosses are independent, hence the probability of our event is the product of $\frac{5}{6}$ five times by itself.

1.7

A store sells three different brands of washers, A, B, C, with a relative frequency of, respectively, $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$. The three brands require warranty repairs during their first year with respective probabilities $0.3, 0.6, 0.5$. For one of these washers, whose brand we do not know - it may be any of the three - the probability of a warranty repair during the first year is (choose one)

○ 0.3556

○ 0.4011

⊗ 0.4333

Discussion: We apply the "total probability" formula. Let's call A, B, C the events "we sold washer A, **B**, or C" respectively. Then $P[A] = \frac{1}{2}, P[B] = \frac{1}{3}, P[C] = \frac{1}{6}$. Let's call R the event "washer needs a warranty repair". We have

$$P[R|A] = 0.3, P[R|B] = 0.6, P[R|C] = 0.5$$

Hence,

$$P[R] = P[R|A]\,P[A]+P[R|B]\,P[B]+P[R|C]\,P[C] = 0.3\cdot\frac{1}{2}+0.6\cdot\frac{1}{3}+0.5\cdot\frac{1}{6} = 0.433\ldots$$

1.8

We throw two dice, and their respective points are X and Y. Let $A = \{X + Y = 7\}, B = \{Y \neq 3\}$. Then (choose one)

⊗ A and B are independent, and $P[A \cap B] = \frac{5}{36}$

◯ A and B are not independent, and $P[A \cap B] = \frac{1}{36}$

◯ A and B are independent, and $P[A \cap B] = \frac{2}{36}$

Discussion: We know from the book and from our discussion in class that A and $\{Y \neq 3\}$ are independent (we showed that A and $\{Y = 3\}$ are independent, but that's the same, since the one is the complement of the other). Since, $P[A] = \frac{1}{6}$, and $P[Y \neq 3] = 1 - P[Y = 3] = 1 - \frac{1}{6} = \frac{5}{6}$, we have the answer. We can also argue directly: $A \cap B$, means that $Y \neq 3$, and the sum of the points is 7. Hence, Y can be 1,2,4,5,6, and X is locked in each case to be equal to $7 - Y$. Hence we have 5 possibilities, each with probability $\frac{1}{36}$, for a total of $\frac{5}{36}$. Since this also happens to be $P[A] \cdot P[Y \neq 3]$, we have proved again that they are independent.

1.9

Consider two events A and B, such that $P[A] = 0.1$, $P[B^c] = 0.8$, $P[A \cap B] = 0.01$. Then (choose one)

◯ $P[A^c \cap B^c] = 0.53$

◯ $P[A^c \cap B^c] = 0.78$

⊗ $P[A^c \cap B^c] = 0.71$

Discussion: Again, we note that $A^c \cap B^c = (A \cup B)^c$. Now, with the data we have, we can easily compute $P[A \cup B] = P[A] + P[B] - P[A \cap B] = 0.1+(1-0.8)-0.01 = 0.29$. Consequently, $P[(A \cup B)^c] = 1-0.29 = 0.71$. We could also compute directly,

$$P[A^c \cap B^c] = P[A^c] + P[B^c] - P[A^c \cup B^c]$$

$$P[A^c \cup B^c] = 1 - P[A \cap B] = 0.99; P[A^c] = 1 - 0.1 = 0.9$$

$$P[A^c \cap B^c] = 0.9 + 0.8 - 0.99 = 0.71$$

1.10

A fair coin is tossed five times, and every time the result is "Heads". The probability that the sixth toss will result in yet another "Head" is (choose one):

○ 0.5^6

⊗ 0.5

○ $\frac{0.5}{6}$

○ $\frac{0.5}{5}$

○ 0.5^5

Discussion: The standard model for coin tossing is that each toss is independent of all others. Hence, the fact that we had heads for 5 consecutive tosses doesn't affect the probabilities of the next toss in the least.

1.11

There are two copiers in an office, a and b. Define the events A="a is working", B="b is working" e C="at most one of the two copiers is *not* working". Which one of the following holds?

○ $C = A^c \cup B^c$

○ $C = A^c \cap B^c$

⊗ $C = A \cup B$

○ $C = (A \cup B)^c$

Discussion: If at most one copier is not working, we have that one copier is working for sure, and the other might possibly be working too. Hence, we are in an "OR" situation: "either a, or b, or, possibly both work": $A \cup B$

2 Problems

2.1

We have two coins, with respective probabilities of coming up heads p_1, and p_2. Assume $0 < p_1 < p_2 < 1$. One of us flips coins in order 1-2-1, and the other flips them in the order 2-1-2. Consider the two events $A =$"the first sequence contains at least two consecutive H", and $B =$"the second sequence contains at least two consecutive T"

1. Express $P[A]$ and $P[B]$.
2. Can you tell which is greater?

Solution: We have that A consists of the sequences HHT, THH, HHH. They have probabilities, $p_1(1-p_1)p_2, p_2(1-p_1)p_1$, and $p_1^2 p_2$. A similar list works for B. Hence.

1. Respectively:

$$P[A] = 2p_1(1-p_1)p_2 + p_1^2 p_2 = 2p_1 p_2 - p_1^2 p_2 = p_1 p_2 (2 - p_1)$$

$$P[B] = (1-p_2)(1-p_1)p_2 + p_2(1-p_1)(1-p_2) + (1-p_2)^2(1-p_1) =$$

$$= 2p_2(1-p_1)(1-p_2) + (1-p_2)^2(1-p_1) = (1-p_1)(1-p_2)(2p_2 + 1 - p_2) =$$

$$= (1-p_1)(1-p_2)(1+p_2) = (1-p_1)\left(1-p_2^2\right) = 1 - p_1 - p_2^2 + p_1 p_2^2$$

2. This is easier than it looks, if you stop and consider the symmetry in the problem. Note that the first run uses 1-2-1 where $p_1 < p_2$, and we are looking for sequences sequences of heads. The second uses 2-1-2, but look at sequences of tails, whose probabilities are $(1-p_2) < (1-p_1)$! Hence, the range we are interested in (the triangle $0 < p_1 < p_2 < 1$ in the unit square of the plane) will split in two equal subsets, one in which $P[A] > P[B]$, and one where the opposite holds. By symmetry again, the dividing line should be the mid-line, i.e. $p_1 = 1 - p_2$, or $p_1 + p_2 = 1$. Check that the limit values confirm this intuition:

$$p_1 = 0 \Rightarrow P[A] = 0; P[B] = 1 - p_2^2$$

$$p_2 = 1 \Rightarrow P[A] = 2p_1 - p_1^2, P[B] = 1 - p_1 - 1 + p_1 = 0$$

$$p_1 + p_2 = 1 \Rightarrow P[A] = P[B]$$

We can also note that on the remaining boundary,

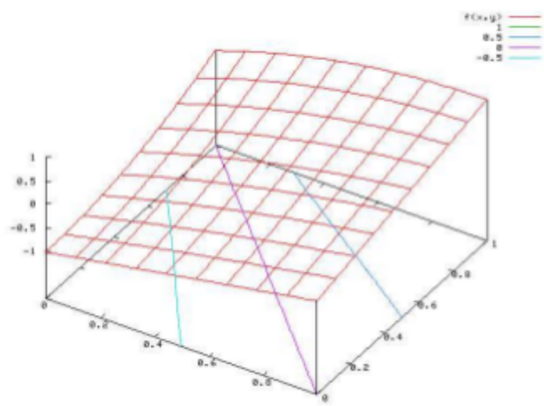
$$p_1 = p_2 \Rightarrow P[A] = 2p^2 - p^3; P[B] = 1 - p - p^2 + p^3 \Rightarrow$$

$$\Rightarrow P[A] - P[B] = 3p^2 - 2p^3 + p - 1 = f(p)$$

Note that

$$f'(p) = 6p - 6p^2 + 1$$

is never zero in $[0, 1]$ (it vanishes at $\frac{1}{2} \pm \frac{\sqrt{3}}{6}$), hence is always positive, and $f(1) = 1, f(0) = -1, f\left(\frac{1}{2}\right) = 0$. This analysis could be repeated for every line $p_1 + x = p_2$, $x > 0$ fixed, and show the same behavior, proving our intuition. Also, to get a visual picture, we could let a computer plot $P[A] - P[B]$ as a function of the two variables p_1, p_2 over our triangle. Here is a plot (by GNUplot) of $P[A] - P[B]$ over the square $0 \leq p_1 \leq 1, 0 \leq p_2 \leq 1$ - we are interested in the triangle $p_1 < p_2$. The lines on the base are contour lines, as indicated in the legend.



2.2

Here is a different story line for the very same problem as in 2.1: Abe and Ben share two bicycles. The first is in working order on any given day with probability p_1, independently of all other days, and the second is, again independently, working with probability p_2. They switch bikes every day, over three days, Abe starting with 1, and Ben starting with 2. Assume $0 < p_1 < p_2 < 1$. Consider the events $A =$"Abe has a non-working bike at least on two consecutive days", and $B =$"Ben has a non-working bike at least on two consecutive days".

1. Express $P[A]$ and $P[B]$.

2. Can you tell whether which is greater?

Solution: see #2.1

2.3

Given three fair six-sided dice, we let X_1, X_2, X_3 denote the points they show after a toss. Determine if the following events are independent:

$$A = \{X_1 > X_2\}, B = \{X_2 > X_3\}$$

Solution: First of all, it is easy to argue that $P[A] = P[B] = \frac{15}{36} = \frac{5}{12}$: the two events must have the same probability, and, moreover, their probability can be computed by counting all pairs (i, j), with $i < j$. In fact, we need to subtract the pairs with $i = j$ (there's 6), and, by symmetry, half of the remaining 30 are as we desire. Hence, $P[A] \cdot P[B] = \frac{25}{144} = .17361$.
Now, let us look at $P[A \cap B] = P[X_1 > X_2 > X_3]$. This is the probability of all triplets (i, j, k) with $i < j < k$, and can be obtained by simple counting again: for $i = 1, j = 2$ we have 4 possible values for k; for $i = 1, j = 3$, we have 3, and so on; for $i = 2, j = 3$ we have 3 possible values for k, and so on, and so on. The result is

$$(4 + 3 + 2 + 1) + (3 + 2 + 1) + (2 + 1) + 1 =$$

$$= 10 + 6 + 3 + 1 = 20$$

out of the $6^3 = 216$ possible triplets. Hence,

$$P[A \cap B] = \frac{20}{216} = \frac{5}{54} = .092593$$

, so the answer is "no!"
We could also count our "favorable cases" in a different way. We look for all triplets with different numbers, regardless of order (since we have a fixed order in mind, we only need to know the entries). One element of this triplet can be chosen in 6 ways, the second in 5, the third in 4, and then we have to divide by the number of possible arrangements of the same three numbers: $\frac{6 \cdot 5 \cdot 4}{3!} = \binom{6}{3} = 20$.
Finally, we could formalize the first calculation by repeated conditioning:

$$P[A \cap B] = P[X_1 > X_2 > X_3] = \sum_{i=1}^{6} P[X_1 > X_2 > X_3 | X_3 = i] P[X_3 = i] =$$

$$\sum_{i=1}^{6} P[X_1 > X_2 > i | X_3 = i] P[X_3 = i] = \sum_{i=1}^{6} P[X_1 > X_2 > i] P[X_3 = i]$$

We have used a not-so-obvious fact (at first glance), due to the fact that statements involving X_1 and X_2 are independent of statements involving

X_3: since we are conditioning on $X_3 = i$, we may substitute i for X_3 in the inequalities, but now X_3 does not appear in the inequalities any more, so the conditional probability is equal to the unconditional probability! Now, $P[X_1 > X_2 > i] = 0$ for $i = 6, 5$ (we are looking for *strict* inequality). Otherwise,

$$P[X_1 > X_2 > i] = P[X_1 > X_2 | X_2 > i] \, P[X_2 > i]$$

$$P[X_1 > X_2 | X_2 > i] = \sum_{k=i+1}^{6} P[X_1 > X_2 | X_2 = k, X_2 > i] \, P[X_2 = k | X_2 > i] =$$

$$\sum_{k=i+1}^{6} P[X_1 > k] \, P[X_2 = k | X_2 > i]$$

where we have used the same argument we used in getting rid of X_3 above. Again, $P[X_1 > 6] = 0$, so the sum actually extends to $k = 5$. Now we can use the numbers

$$P[X_3 = i] = \frac{1}{6}$$

$$P[X_2 > 1] = \frac{5}{6}; P[X_2 > 2] = \frac{4}{6}; P[X_2 > 3] = \frac{3}{6}; P[X_2 > 4] = \frac{2}{6}$$

$$P[X_2 = k | X_2 > i] = \frac{P[X_2 = k]}{P[X_2 > i]} = \frac{1}{6} \cdot \frac{1}{P[X_2 > i]} \qquad (k > i)$$

(since the conditional probability is zero when $k \leq i$). Also,

$$P[X_1 > 2] = \frac{4}{6}; P[X_1 > 3] = \frac{3}{6}; P[X_1 > 4] = \frac{2}{6}; P[X_1 > 5] = \frac{1}{6}$$

Combine them all together:

$$P[A \cap B] = \sum_{i=1}^{4} P[X_3 = i] \sum_{k=i+1}^{5} P[X_1 > k] \, P[X_2 = k | X_2 > i] \, P[X_2 > i] =$$

$$= \frac{1}{6} \left\{ \left(\frac{4}{6} + \frac{3}{6} + \frac{2}{6} + \frac{1}{6} \right) \cdot \frac{1}{6} + \left(\frac{3}{6} + \frac{2}{6} + \frac{1}{6} \right) \cdot \frac{1}{6} + \left(\frac{2}{6} + \frac{1}{6} \right) \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} \right\} =$$

$$= \frac{1}{216} (4 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1) = \frac{1}{216} (4 + 12 + 4) = \frac{20}{216} = \frac{5}{54}$$

Remark Sometimes, the expression $\frac{P[A \cap B] - P[A]P[B]}{\sqrt{P[A]P[A^c]}\sqrt{P[B]P[B^c]}}$ is called the "correlation" between the two events: if they are independent, it is equal to 0 (note that the denominator is the product of the "geometric means" of the probabilities of A and B and their complements: it serves as a normalizing factor, as we will better understand in the sequel). In our case, it is equal to

$$\frac{\frac{5}{54} - \frac{25}{144}}{\sqrt{\frac{5}{12} \cdot \frac{7}{12}} \cdot \sqrt{\frac{5}{12} \cdot \frac{7}{12}}} = \frac{\frac{5}{54} - \frac{25}{144}}{\frac{35}{144}} = -\frac{1}{3}$$

indicating some dependence, as we knew, and as could have been expected from the start (correlations vary between -1 and 1).

2.4

A manufacturer has a well-established 2.5% probability of producing a defective piece of equipment. Each piece is screened for defects before shipping, and the procedure is known to catch defective pieces at a 98% rate, as well as erroneously flagging 3% of good pieces. A random piece of equipment arrives for screening:

1. What is the probability that it will be flagged?
2. What is the probability that the piece is defective *and* it is flagged?
3. What is the probability that a flagged item is, in fact, defective?

Solution: You will recognize the structure of the problem: it is the "imperfect diagnosis test" problem. Hence, the solution is the same:

1. We have $P[D] = 0.025$ for a piece being defective, and $P[F|D] = .98$, $P[F|D^c] = .03$ for a piece being flagged, depending on its status. Thus (total probability):

   $$P[F] = P[F|D]\,P[D] + P[F|D^c]\,P[D^c] = .98 \cdot .025 + .03 \cdot .975 = .05375$$

2. Here we are looking at the probability of the intersection, *not* the conditional probability:

   $$P[F \cap D] = P[F|D]\,P[D] = .98 \cdot .025 = .0245$$

3. This is the classical Bayes' Rule application:

   $$P[D|F] = \frac{P[F|D]\,P[D]}{P[F]} = \frac{.0245}{.05375} = 0.45581$$

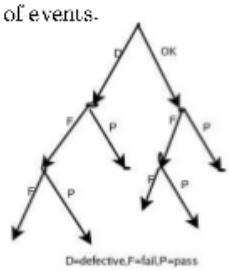
   (as usual, the rarity of actual defective pieces makes the diagnosis problematic).

2.5

The same manufacturer as in problem 2.4 has now adopted the following protocol: when a piece is flagged as defective, it is tested again. If the result is still "defective", the piece is discarded, while it is shipped in every other case.

1. What is the probability that a random piece of equipment will be discarded?
2. What is the probability that a discarded piece is, in fact, defective?
3. What is the probability that a defective item will be shipped anyway?

Solution: We can use the numbers we found in problem 2.4. We know that a piece is discarded if it is flagged twice. Consider a graph for the sequence of events.



The first time, the piece is flagged with probability .05375. Now, its *conditional* probability of being defective is .45581.

1. Hence, the probability of being discarded is

$$P[F_2F_1] = P[F_2|F_1]\,P[F_1]$$

$$P[F_2|F_1] = P[F_2|DF_1]\,P[D|F_1] + P[F_2|D^cF_1]\,P[D^c|F_1]$$

and

$$P[F_2F_1] = (P[F_2|DF_1]\,P[D|F_1] + P[F_2|D^cF_1]\,P[D^c|F_1])\,P[F_1] = \\ = (.98 \cdot .45581 + .03 \cdot .54419) \cdot .05375 = .024887 \tag{1}$$

2. We are asking for $P[D|F_1F_2]$. By Bayes,

$$P[D|F_1F_2] = \frac{P[F_2|DF_1]\,P[DF_1]}{P[F_2F_1]}$$

The numerator is the first half of the expression (1):

$$.98 \cdot .45581 \cdot .05375 = .02401$$

and the result is

$$\frac{.02401}{.024887} = .96475$$

- a marked improvement over the one-test odds!

3. This is $P[F_1F_2|D]$. Note that, even though the two tests are given "independently", F_1 and F_2 are not independent events, since they refer to the same piece: knowledge that F_1 occurred, markedly changes the probability that F_2 will occur too! Ignoring this small element causes some mistakes in "double-testing" calculations. However, they are *conditionally* independent, if we *know* the "status" of the patient. Applying the definitions,

$$P[F_1F_2|D] = P[F_2|DF_1]\,P[F_1|D] = .98 \cdot .98 = .9604$$

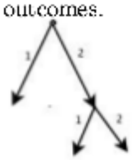
In other words, with this more expensive, but more cautious, protocol, 96% of the defective items will be caught, and there is less than a 4% probability that a good item will be discarded (that's from point 2).

2.6

Solve the problem that tradition suggests is at the origin of Probability Theory:

> Two players play a game of pure chance, with probability p of winning for player #1 (you can reduce to $p = 0.5$, the original, if you wish). They match up in a best-of-seven series (first player to win 4 games is the overall winner). A pot of $100 is at stake. Circumstances force them to interrupt the match when player #1 is ahead 3-2. What is the fair[1] way to split the pot between the two? 100-0, 50-50, 60-40 (in the case $p = 0.5$), <put your own suggestion here>?

Solution: The argument adopted by the founding fathers of probability was to compute the probability of either of the two players winning the overall match, assuming each game is independent of the rest, and is played with the same p. Of course, a graph is a good way of listing the possible outcomes.



Letting "1" denote a win by player 1, and "2" a win by player 2, we have, overall,

$$1; 21; 22$$

I.e., conditioning on the current score (3-2), we have that the match will end in favor of 1 in the first two sequences, and of 2 in the third. The first sequence has probability p, the second $p(1-p)$, and the third $(1-p)^2$. Hence, the probability of #1 winning the match is

$$\frac{p + p(1-p)}{p + p(1-p) + (1-p)^2} = \frac{2p - p^2}{2p - p^2 + 1 - 2p + p^2} = p(2-p)$$

If $p = \frac{1}{2}$, this is equal to $\frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$, and the pot should be split $3:1$.

[1] We all agree that "fairness" is a subjective quality. However, you are supposed to interpret the term in the same spirit that a game that pays $1 on a $1 bet, and has even chances of being won or lost, is "fair".

2.7

Here is a sampling of a statistical argument. A service (say, a printer) receives requests from two different departments at regular intervals. Given the size of the departments, it is reasonable to assume that proper use of the printer would have p of the requests be from department #1, and $1-p$ from department #2. To verify that none of the departments is improperly hogging the printer, a log of the requests is made over a day, and results in the following sequence:

$$1 \quad 1 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1 \quad 2 \quad 1$$

1. What is the probability of obtaining this sequence if both departments are operating properly?
2. If you the exact sequence is not recorded in the log, but only the number of requests by each department over the day, what is the probability of observing the previous outcome (i.e., 6 requests from #1, and 4 from #2)?
3. Suppose now that $p = 0.1$ (i.e., Dept. #1 is supposed to ask for the printer only 10% of the time). What is the probability of our log (under either method)? What is the probability that the number of calls by Dept. #1 would be greater or equal to 6, out of 10?

Solution:

1. We have 6 calls from Dept. #1, and 4 from Dept. #2, so this sequence has probability $p^6(1-p)^4$.
2. Now we don't have the precise sequence, so the probability we are being asked is the probability of the event "6 calls from #1, and 4 from #2", which is made up of all sequences with the given number of calls: $\binom{10}{6} p^6(1-p)^4 = 210p^6(1-p)^4$
3. The probability of 6 *or more* calls is

$$\sum_{i=6}^{10} \binom{10}{i} p^i (1-p)^{10-i}$$

   With a good calculator, setting $p = 0.1$, we get $1.469 \cdot 10^{-4}$

Remark: The logic of this last calculation is to check that the event we observed has a very low probability, so one possible explanation would be that the model used (i.e., $p = 0.1$) is not appropriate - of course, since the probability is still positive, this is not *proof* that the model is wrong, but only *evidence*.