About Independence

Math 394

The book does not address the notion of independence for random variables until Chapter 6, that is beyond the chapters that we cover in this class. However, the notion is simple enough, and useful enough to let us talk about it already.

Definition: *Consider a finite collection of discrete random variables $X_1, X_2, \ldots, X_n$. They are said to be independent if $P[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n] = P[X_1 = x_1]\, P[X_2 = x_2] \ldots P[X_n = x_n]$, for any choice of $x_1, x_2, \ldots, x_n$. In other words, if the events $\{X_k = x_k\}$ are all independent of each other.*

This connects to our definition of independence for events: if the X_k are indicator functions of events, as in $X_k = 1_{A_k}$, we are requesting that all combinations of A_k's and their complements be independent. In fact, the request is that $P[1_{A_1} = x_1, 1_{A_2} = x_2, \ldots, 1_{A_n} = x_n] = P[1_{A_1} = x_1]\, P[1_{A_2} = x_2] \ldots P[1_{A_n} = x_n]$, where now all the x_k are either 0 or 1.

An easy fact (the proof is similar to the one we saw for events) is that if a collection of n random variables satisfies the definition above, so does any sub-collection of the same random variables.

Definition: *Consider a collection of infinitely many random variables. They are said to be independent if any finite sub-collection is independent according to the definition above.*

A typical example of independent random variables is given by a sequence of "Bernoulli trials": $X_1, X_2, \ldots$with each $X_k$being a Bernoulli random variable with parameter p, and all being independent of each other.

The intuitive meaning of independence of random variables is, as for events, the idea that knowledge of the value that was observed for some, does not change the probabilities of observing values for the others (the fact that a number of coins turned up heads, does not change the probability of heads for any other coin). However, bear in mind that independence is a technical concept, and while intuition can be used to guide us in determining whether a real-life situation can be reasonably modeled with independent random variables, given a specific probabilistic model, independence (or lack thereof) must be proven, and cannot be assumed!