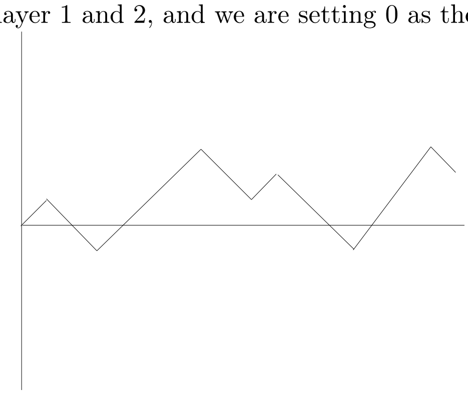
Gambler's Ruin

Math 394

1 The General Problem

Suppose two players play consecutive games, where player #1 as probability p of winning any game, independently of the others, and player #2 has winning probability $1-p = q$. Suppose also that player #1 starts with a capital of c_1, and player #2 with a capital of c_2. Each time a player wins one unit of the other's capital moves to his. In the following schematic picture, you can follow one possible movement of $c - (c_1 - c_2)$, where c is the difference in capital between player 1 and 2, and we are setting 0 as the starting point:



The problem is to find the probability that one or the other will lose all her capital at some point.[1]

2 The "Fair" Case (from T.A. Rozanov: *Probability Theory: A Concise Course*, Dover)

When $p = q = \frac{1}{2}$, we can solve the problem as follows. Let's look at player #1, she starts at level c_1, and she wins if she reaches a level of $c_1 + c_2 = m$ before reaching level 0, and loses if she reaches 0 efore reaching level m. We will find the probability of ruin, as a function of the initial capital: her capital at time 0 is c_1, but at every game, it is as if the match started anew, with player #1

[1] This is the historical origin of this general model. If you think instead of a "particle" moving left or right with assigned probabilities, each step being independent of the previous ones, we have a "random walk", also affectionately known as as the "drunkard's walk". You may have heard how small variations of this model are very popular on Wall Street as models of the stock market.

having a capital of x, as determined by the preceding games. Let $\pi(x)$ be the probability of ruin starting with x dollars. We have two possibilities: player #1 wins the first game, and her probability of ruin is now $\pi(x+1)$, or she loses her first game, so her probability of ruin is $\pi(x-1)$. If A is the event "player #1 will eventually lose"), and W and L are the events, respectively, that she wins or loses the first game, we have

$$P[A] = P[A|W]\,P[W] + P[A|L]\,P[L]$$

Starting with x dollars, this means

$$\pi(x) = \frac{1}{2}\left[\pi(x+1) + \pi(x-1)\right] \tag{1}$$

with

$$\pi(0) = 1, \pi(m) = 0 \tag{2}$$

It should be obvious that equation (1) implies that $\pi(x)$ is a linear function, $\pi(x) = ax + b$, and the boundary conditions (2) imply that $b = 1$, $am + b = 0$, i.e. $a = -\frac{1}{m}$:

$$\pi(x) = 1 - \frac{x}{m}$$

At the start of the game,

$$\pi(c_1) = 1 - \frac{c_1}{c_1 + c_2}$$

An interesting consequence is that if $c_2 \to \infty$ (player #2 has an essentially limitless ability to handle losses), $\pi(c_1) \approx 1$.

Since this holds true, no matter how large (but fixed) c_1 is, we conclude that our graph (the random walk graph), will eventually plunge as far into negative territory as we wish. By symmetry, of course, it will also climb as far into positive territory as we wish!

3 Asymmetric Games

Suppose now that $p \neq \frac{1}{2}$. Instead of equation (1), we will now have

$$\pi(x) = p \cdot \pi(x+1) + (1-p) \cdot \pi(x-1) = p \cdot \pi(x+1) + q \cdot \pi(x-1)$$

Since $p + q = 1$, we can also write

$$p\left(\pi(x+1) - \pi(x)\right) = q\left(\pi(x) - \pi(x-1)\right)$$

$$\frac{\pi(x+1) - \pi(x)}{\pi(x) - \pi(x-1)} = \frac{q}{p}$$

with the same boundary conditions (2). With a little induction, starting from $x = 1$, and $\pi(0) = 1$, we end up with

$$\frac{\pi(x+1) - \pi(x)}{\pi(1) - 1} = \left(\frac{q}{p}\right)^x$$

In particular,

$$\pi(x{+}1){-}\pi(0) = \sum_{k=0}^{x} (\pi(k+1) - \pi(k)) = (\pi(1) - 1) \sum_{k=0}^{x} \left(\frac{q}{p}\right)^k = (\pi(1) - 1)\, \frac{1 - \left(\frac{q}{p}\right)^{x+1}}{1 - \frac{q}{p}} \tag{3}$$

We now impose the second boundary condition (2), setting $x = m - 1$ in (3)

$$\pi(m){-}\pi(0) = \sum_{x=0}^{m-1} (\pi(x+1) - \pi(x)) = (\pi(1) - 1) \sum_{x=0}^{m-1} \left(\frac{q}{p}\right)^x = (\pi(1) - 1)\, \frac{1 - \left(\frac{q}{p}\right)^{m}}{1 - \frac{q}{p}}$$

and since$\pi(m) = 0$

$$-1 = \pi(1) \frac{1 - \left(\frac{q}{p}\right)^{m}}{1 - \frac{q}{p}} - \frac{1 - \left(\frac{q}{p}\right)^{m}}{1 - \frac{q}{p}}$$

$$\pi(1) = \left(\frac{1 - \left(\frac{q}{p}\right)^{m}}{1 - \frac{q}{p}} - 1 \right) \frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^{m}} = 1 - \frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^{m}} \tag{4}$$

and

Combining (3), and (4), we finally find

$$\pi(x) = 1 - \frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^{m}} \cdot \frac{1 - \left(\frac{q}{p}\right)^{x}}{1 - \frac{q}{p}} = 1 - \frac{1 - \left(\frac{q}{p}\right)^{x}}{1 - \left(\frac{q}{p}\right)^{m}} =$$

$$\frac{1 - \left(\frac{q}{p}\right)^{m} - 1 + \left(\frac{q}{p}\right)^{x}}{1 - \left(\frac{q}{p}\right)^{m}} = \frac{\left(\frac{p}{q}\right)^{m-x} - 1}{\left(\frac{p}{q}\right)^{m} - 1}$$

(having multiplied numerator and denominator by $\left(\frac{p}{q}\right)^{m}$)

Summing up, we have that the probability of ruin of player #1 is

$$\pi\left(c_1\right) = \frac{1 - \left(\frac{p}{q}\right)^{c_2}}{1 - \left(\frac{p}{q}\right)^{m}}$$

and, by reversing roles, it is easy to conclude that the probability of ruin for player #2 is

$$1 - \pi\left(c_1\right) = \frac{1 - \left(\frac{q}{p}\right)^{c_1}}{1 - \left(\frac{q}{p}\right)^{m}}$$

(in other words, even though it would be theoretically possible for neither player ever being ruined, this event has probability zero).

This result has interesting consequences. Suppose $c_2 \to \infty$, while c_1 stays fixed. If $p < q$, clearly $\pi(c_1) \to 1$. However, if $p > q$,

$$1 - \pi(c_1) \to 1 - \left(\frac{q}{p}\right)^{c_1} \qquad \pi(c_1) \approx \left(\frac{q}{p}\right)^{c_1}$$

In other words, if player #1 has an advantage (e.g., she is a better player), even in the face of an opponent of almost unlimited resources, has a fighting chance to avoid ruin.

In yet other words, the graph we started with, describing a random walk, would, in this case, eventually climb higher than any pre-defined bound, while reaching a lower limit (which is random), that it will never cross down (consider $c_1 \to \infty$), eventually climbing towards $+\infty$.