A Random Collection of Problems

Math 394

Quizzes

1

Suppose A and B are such that $P[A] = .6, P[B] = .8, P[A|B] = .375$. Then, $P[A \cup B] =$

- ○ .6
- ○ .8
- ○ .95
- ○ None of the above - the system is not coherent

2

Consider n independent Bernoulli RVs, all with parameter p. The distribution of the RV $Y = X_1 \cdot X_2 \cdot \ldots \cdot X_n$ is

- ○ Binomial $bin(n, p)$
- ○ The same as the distribution of e^{N_n}, where N_n is a binomial $bin(n, p)$
- ○ $P[Y = k] = p^k \quad k = 0, 1, \ldots, n$
- ○ $P[Y = k] = \frac{p^k}{(1-p)^{1-k}} \quad k = 0, 1$
- ○ $P[Y = k] = p^n \; k = 1, \quad P[Y = k] = 1 - p^n \; k = 0$
- ○ $P[Y = k] = p^n \; k = 1, \quad P[Y = k] = (1 - p)^n \; k = 0$

3

Recalling the definition of variance for a RV, what is the *general* formula for $Var[X - Y]$, where X, and Y are two general RVs (no special assumptions are made, except that their first and second moments are all well defined). You have your book, so you shouldn't need this, but, anyway, here goes:

$$Var[X] = E[X^2] - (E[X])^2$$

$$Cov[X, Y] = E[XY] - E[X]E[Y]$$

- ○ $Var[X] + Var[Y]$
- ○ $Var[X] - Var[Y]$
- ○ $Var[X] + Var[X] + Cov[X, Y]$
- ○ $Var[X] + Var[X] - Cov[X, Y]$
- ○ $Var[X] + Var[X] + 2Cov[X, Y]$
- ○ $Var[X] + Var[X] - 2Cov[X, Y]$

4

Write a consequence that follows from the following statements:

If A and B are independent, then...

If $A \subseteq B$, then $(P[A|B] =, P[B|A] =, P[A \cup B] =, P[A \cap B] = ...)$

Suppose $X_1, X_2, \ldots, X_n$ are independent Bernoulli RVs. Then the distribution of

$X_1^2 + X_2^2 + \ldots + X_n^2$ is...

$e^{X_1} + e^{X_2} + \ldots + e^{X_n}$ is...

$e^{X_1+X_2+\ldots+X_n}$ is...

Let X be a standard normal RV. Then

$P[|X| < 1] =$

$P[|X| < 2] =$

$P[|X| > 2] = ...$

5

If A and B are independent events, then (there could be more than one correct answer)

- ○ $P[A \cup B] = P[A] + P[B]$
- ○ $P[A \cap B] = P[A]P[B]$
- ○ $P[A \setminus B] = P[A] - P[B]$
- ○ $P[A] = 0$, or $P[B] = 0$, or both

6

If X is a RV with binomial distribution with parameters (n, p), then $P[nX = k] =$(there could be more than one correct answer)

- ○ $\binom{n^2}{k} p^k (1-p)^{n-k}$
- ○ $\binom{n}{nk} p^{nk} (1-p)^{n-nk}$
- ○ $\binom{n}{\frac{k}{n}} p^{\frac{k}{n}} (1-p)^{n-\frac{k}{n}}$
- ○ $\binom{\frac{n}{k}}{k} p^k (1-p)^{\frac{n}{k}-k}$
- ○ $\binom{n^2}{nk} p^{nk} (1-p)^{n^2-nk}$
- ○ $\binom{n}{k} p^k (1-p)^{n-k}$

7

Suppose X is a normally distributed RV, with $\mu = 1, \sigma^2 = 2$, then (there could be more than one correct answer)

- ○ $P[X < 1] = P[X > 1]$
- ○ $P[X < 0] = P[X > 0]$
- ○ $P[X < -2] = P[X > 2]$
- ○ $P[X < 2] = P[X > 2]$

Problems

1

A test for defects in a shipment is performed on a sample, and this procedure is known, from statistical theory, to have a probability of 5% of wrongly rejecting the shipment, and a probability of 10% of wrongly accepting the shipment. The original manufacturer is known to have a 2% rate of defective shipments.

1. When the test result is negative (hence, the shipment is accepted), what is the probability that the shipment is actually defective?

2. When the test result is positive (hence, the shipment is rejected), what is the probability that the shipment is indeed defective?

3. If the procedure is amended to include a second test (independent, but with the same correctness as the first one) when the test is failed, and the shipment is rejected only if both tests are positive, what are the probabilities that a good shipment will be rejected, and that a bad shipment will be accepted?

2

Suppose X_j $j = 1, 2, \ldots$ are RVs with distribution

$$P[X_j = 1] = p; P[X_j = -1] = 1 - p$$

and that they are independent. Let $S_n = \sum_{k=1}^{n} X_k$

1. Write the pmf of S_n

2. Write $E[S_n], Var[S_n]$, and the approximate distribution (e.g., the cdf, or something equivalent) for $\frac{S_n - ES_n}{\sqrt{Var[S_n]}}$ when n is very large.

3. Assuming now that the parameter p is of the form $p_n = \frac{p}{n}$, for some number p, write the approximate pmf for S_n, when n is very large.
   Note: be very careful here! You need to adjust what you can find in the book, in order to fit the present situation.

4. Suppose $n = 1000$, $p = .3$. Use one of the results above to evaluate explicitly $P[S_n \leq 500]$

3

Suppose U is a uniform RV, with $P[0 \leq U \leq 1] = 1$. Let $g(x) = \log \frac{1}{x}$. Write the distribution of $g(U)$.
Hint: compute the cdf or the survival function of $g(U)$

4

A RV has density

$$f_X = \begin{cases} cx(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1. What is the value of c?

2. What is the expected value of X?

3. What is the variance of X?

5

Two theories are competing as explanation of a phenomenon. One assumes that when input $X = n$ (n is an integer), the output Y should be an exponential RV with parameter $\frac{1}{n}$. The other assumes that, for the same input, the output should be Gaussian with parameters $\mu = n$, $\sigma^2 = n^2$.

1. How would the two theories differ in predicting the likelihood of an output in the range $Y > 2n$, for $X = n$.
2. A blind experiment is performed and an output of $Y \in (9, 12)$. Using tables and calculator, find the conditional probability for $X = 10$, according to both models.

6

Let X_i, $i = 1, 2, \ldots, 10$ all take values $0, \pm 1$, each with probability $\frac{1}{3}$.

1. What are the possible values for $S = \sum_{i=1}^{10} X_i$?
2. Assuming that all X_i are independent, what are the probabilities $P[S = 10]$, and $P[S = -10]$?
3. Again assuming independence, what is the probability $P[S = 9]$?

7

A binary signal is transmitted, each bit having, independently, a 10% chance of being received wrongly.

1. If each packet is composed of 10 bits, what is the distribution of the number of errors in the reception?
2. What is the probability that a packet arrives corrupted (i.e., not all bits are received correctly)?
3. Suppose the 10th bit is a "parity" bit, i.e. a 1 of the other 9 bits have an odd number of 1s, and a 0 if they have an even number (0 is even). What is the probability that a corrupted packet will go undetected?

8

A rat in a maze is confronted by a sequence of forks, where it has to choose between going right or left. To test its sense of smell, a piece of cheese is placed at the left end of a 20-fork maze, and its choices are recorded, in order to see if they show a left-ward bias, compared to a no-cheese maze.

The results are:

- no-cheese: 12 left turns, 8 right turns
- cheese: 14 left turns, 6 right turns

1. Assuming no bias is really present, i.e. the probabilities of a left or right turn are $p = \frac{1}{2}$, what are the probabilities that the rat would make a number of left turns, $n \geq 12$, and $n \geq 14$? Write an exact formula, and use an approximation to get an explicit number.
2. The theory of the behavioral scientists was that the difference between the two setups (left-right), would have been distributed (in the Gaussian approximation) with a $\mu = 5$, $\sigma = 2$ (note: that is σ, not σ^2). If that model was correct, what would be the probability that we would observe a difference (like we did) $D \leq 2$?

9

A simple model of reliability for an electronic equipment would be the following:

- With probability $p = .01$ the equipment is DOA, i.e. is non operative from the beginning
- Assuming it is *not* DOA, it starts with a *hazard function* of the form

  $$h_1(t) = 1 - t; 0 \le t \le 0.5$$

  then changes to

  $$h_2(t) = 0.5; 0.5 \le t \le 2$$

  and finally ends with

  $$h_3(t) = 0.5 + (t - 2); t \ge 2$$

1. Write the cdf for the lifetime of the equipment.
2. How likely is the equipment going to fail between $t = 0$ and $t = 1$?
3. Assume the component comes with a full warranty up to time $t = 1$. If the cost of replacement is $\$C = \$1,000$, what is the expected cost of this warranty for the manufacturer?
4. *Assuming the component has not failed before* $t = 1$, an extended warranty is offered for full replacement at $1,000, for the time period $1 \le t \le 3$: what is the expected cost of this for the warranty company?