Math 394B Summer 2010

Solutions to Homework Assignment

due 7/26

Problems from Chapter 4

#1

Since X denotes our winning, we have to consider all possible outcomes, and these are (using the initial of the color of the two balls extracted):

- WW $X = -2$
- WO $X = -1$
- WB $X = 1$
- OO $X = 0$
- OB $X = 2$
- BB $X = 4$

As for the respective probabilities, we have a total of 14 balls, and the probabilities are computed remembering that this is an extraction without reinsertion:

- $P[WW] = P[X = -2] = \frac{8}{14} \cdot \frac{7}{13} = \frac{4}{13} = \frac{28}{91}$
- $P[WO] = P[X = -1] = 2 \cdot \frac{8 \cdot 2}{14 \cdot 13} = \frac{16}{91}$
- $P[WB] = P[X = 1] = 2 \cdot \frac{8 \cdot 4}{14 \cdot 13} = \frac{32}{91}$
- $P[OO] = P[X = 0] = \frac{2}{14} \cdot \frac{1}{13} = \frac{1}{91}$
- $P[OB] = P[X = 2] = 2 \cdot \frac{2 \cdot 4}{14 \cdot 13} = \frac{8}{91}$
- $P[BB] = P[X = 4] = \frac{4}{14} \cdot \frac{3}{13} = \frac{6}{91}$

Hence, the probability of losing is $\frac{44}{91}$, of being even $\frac{1}{91}$, and of ending ahead $\frac{46}{91}$, while the expected payoff is

$$EX = \frac{-56 - 16 + 32 + 16 + 24}{91} = 0$$

(The game is considered "fair") Of course, even if the probability of winning is greater than that of losing, that is not enough to ensure a positive *expected payoff* . The amounts involved play a decisive role.

#2 (C)

We call the points on the two dice X and Y - they are obviously RV's and $P[X=i]=P[X=j]=\frac{1}{6}$ for all $i,j,=1,2,\ldots,6$. Let us develop a general formula for the distribution of the product of two *independent* discrete RV's:

$$P[XY=z]=\sum_y P\left[XY=z\,|Y=y\right]P\left[Y=y\right]$$

where we sum over all possible values of Y. Now, using a trick that was already introduced in the solutions to the "practice problems" for the 1st Midterm, we get

$$=\sum_y P\left[Xy=z\,|Y=y\right]P\left[Y=y\right]=\sum_y P\left[X=\frac{z}{y}\,|Y=y\right]P\left[Y=y\right]=$$

$$\sum_y P\left[X=\frac{z}{y}\right]P\left[Y=y\right]$$

where, by independence of X and Y, once we eliminated any reference to the RV Y, by substituting its value on $Y=y$, X does not care any more about being conditioned on the value of Y!

In our specific case, since X and Y take only integer values, we are restricted to z and y, such that $\frac{z}{y}$ is an integer between 1 and 6. Hence,

1. $P\left[XY=1\right]=\sum_y P\left[XY=1\,|Y=y\right]P\left[Y=y\right]=$
   $=P\left[X=1\right]P\left[Y=1\right]=\frac{1}{36}$
2. $P\left[XY=2\right]=P\left[X=2\right]P\left[Y=1\right]+P\left[X=1\right]P\left[Y=2\right]=\frac{2}{36}=\frac{1}{18}$
3. $P\left[XY=3\right]=P\left[X=3\right]P\left[Y=1\right]+P\left[X=1\right]P\left[Y=3\right]=\frac{1}{18}$
4. $P\left[XY=4\right]=P\left[X=4\right]P\left[Y=1\right]+P\left[X=2\right]P\left[Y=2\right]+P\left[X=1\right]P\left[Y=4\right]=$
   $=\frac{3}{36}=\frac{1}{12}$
5. $P\left[XY=5\right]=P\left[X=5\right]P\left[Y=1\right]+P\left[X=1\right]P\left[Y=5\right]=\frac{1}{18}$
6. $P\left[XY=6\right]=P\left[X=6\right]P\left[Y=1\right]+P\left[X=3\right]P\left[Y=2\right]+$
   $+P\left[X=2\right]P\left[Y=3\right]+P\left[X=1\right]P\left[Y=6\right]=\frac{4}{36}=\frac{1}{9}$
7. $P\left[XY=7\right]=0$
8. $P\left[XY=8\right]=P\left[X=4\right]P\left[Y=2\right]+P\left[X=2\right]P\left[X=4\right]=\frac{1}{18}$
9. $P\left[XY=9\right]=P\left[X=3\right]P\left[X=3\right]=\frac{1}{36}$
10. $P\left[XY=10\right]=P\left[X=5\right]P\left[Y=2\right]+P\left[X=2\right]P\left[Y=5\right]=\frac{1}{18}$
11. $P\left[XY=11\right]=0$

12. $P[XY = 12] = P[X = 6]\,P[Y = 2] + P[X = 4]\,P[Y = 3] +$
$+P[X = 3]\,P[Y = 4] + P[X = 2]\,P[Y = 6] = \frac{1}{9}$

13. 0

14. 0

15. $\frac{1}{18}$

16. $\frac{1}{36}$

17. 0

18. $\frac{1}{18}$

19. 0

20. $\frac{1}{18}$

21. 0

22. 0

23. 0

24. $\frac{1}{18}$

25. $\frac{1}{36}$

26. 0

27. 0

28. 0

29. 0

30. $\frac{1}{18}$

31. 0

32. 0

33. 0

34. 0

35. 0

36. $\frac{1}{36}$

#4

Of course, the values taken by X are $1, 2, 3, 4, 5, 6$, since the worst possible case is when all men are ranked ahead of all women, in which case the 6th ranked is necessarily a woman...

Also, it is immediate that $P[X = 1] = 0.5$, since, by symmetry, the highest ranking person will be a man or a woman with equal probability. If $X > 1$, then the highest ranked person is a man, and

$$P[X = 2|X > 1] = \frac{5}{9}$$

(we are picking at random among 9 people, of whom 5 are women), and

$$P[X = 2] = P[X = 2|X > 1]P[X > 1] = \frac{5}{9} \cdot \frac{1}{2} = \frac{5}{18}$$

We can proceed recursively: we now know $P[X > 2] = 1 - \left(\frac{1}{2} + \frac{5}{18}\right) = \frac{4}{18} = \frac{2}{9}$, so that

$$P[X = 3] = P[X = 3|X > 2]P[X > 2] = \frac{5}{8} \cdot \frac{2}{9} = \frac{5}{36}$$

and continuing,

$$P[X = 4] = P[X = 4|X > 3]P[X > 3] = \frac{5}{7} \cdot \left(\frac{2}{9} - \frac{5}{36}\right) = \frac{5}{7} \cdot \frac{3}{36} = \frac{5}{84}$$

$$P[X = 5] = P[X = 5|X > 4]P[X > 4] = \frac{5}{6} \cdot \left(\frac{1}{12} - \frac{5}{84}\right) = \frac{5}{6} \cdot \frac{2}{84} = \frac{5}{252}$$

$$P[X = 6] = \frac{5!5!}{10!} = \frac{1}{252}$$

In the last calculations we took the alternative route of considering all 10! possible arrangements, and the 5! possible ways in which five men could have been put in positions 1-5, and the 5! arrangements in which five women could be put in positions 6-10. Of course, a similar calculation could have been done on the previous cases, just like the last case could have been solved like we solved the previous ones.

#7 (C)

Even if we are concentrating on the particular case of two dice, the point here is we are looking at certain functions of *two independent random variables*, for their distribution. If X and Y are independent RV, we have the following:

1. $P[\max\{X, Y\} \leq z] = P[X \leq z \cap Y \leq z]$, and, by independence, this is the same as

   $$P[X \leq z] P[Y \leq z] = F_X(z) F_Y(z)$$

   where F_X is the cumulative distribution function of X

2. $P[\min\{X,Y\} > z] = P[X > z \cap Y > z]$, and, by independence, this is the same as

$$P[X > z]\,P[Y > z] = R_X(z)\,R_Y(z)$$

where R_X is the "survival function" of X.

3. $P[X+Y=z] = \sum_y P[X+Y=z\,|Y=y]\,P[Y=y] = \sum_y P[X+y=z\,|Y=y]\,P[Y=y]$(by the total probability theorem). By independence, now,

$$= \sum_y P[X+y=z]\,P[Y=y] = \sum_y P[X=z-y]\,P[Y=y]$$

This expression is sometimes called the "convolution product" of the two distributions, and written $p_X * p_Y(z) = \sum_y p_X(z-y)\,p_Y(y)$. You can check that $p_X * p_Y = p_Y * p_X$

4. As for $X - Y$, since this is just $X + (-Y)$, its distribution is given by $p_X * p_{-Y} = \sum_y p_X(z+y)\,p_Y(y)$

Note that these formulas are true for any pair of independent discrete RV's, and, in fact, with the exception of points 3 and 4 (where some care needs to be taken for continuous RV's, as we will see in due time), for any pair of independent RVs at all.

Of course, in our special case, we can find the distributions "by hand", given the small size of the problem, even though the problem only asks for the possible values. In any case, using the formulas above,

a $\max\{X,Y\}$ can take values $1,2,\ldots,6$. Note that, for a die, $P[X \le k] = \frac{k}{6}$, so $P[\max\{X,Y\} \le k] = \frac{k^2}{36}$

b $\min\{X,Y\}$ can likewise take all 6 values, $1,2,\ldots,6$. Since $P[X > k] = 1-\frac{k}{6}$, $P[\min\{X,Y\} > k] = P[\min\{X,Y\} \ge k+1] = \left(1-\frac{k}{6}\right)^2 = \frac{k^2}{36} - \frac{k}{3} + 1 = \frac{k(k-12)}{36} + 1 = 1 - \frac{k}{6}\cdot\frac{12-k}{6}$(here $k = 0,1,\ldots 5$).

c $X+Y$ takes, as we well know, all values $2,3,\ldots,11,12$, and we have referred to this distribution many times: for $k = 2,3,\ldots 7$, the probability is $\frac{k-1}{36}$, and for $k = 7,8,9,\ldots 12$, it is $\frac{12-(k-1)}{36}$

d $X - Y$ takes the values from $1-6=-5$ to $6-1=5$, and all integers in between:

$$P[X-Y=k] = \sum_{j=1}^{6} p_X(k+j)\,p_Y(j)$$

where the sum is automatically limited to values of j such that $6 \ge k+j \ge 1$. Thus for each k we count how many such js there are, and we add $\frac{1}{36}$ for each of them: just by listing the values allowed, we see that for $k \le 0$, $j = |k+1|,\ldots 6$, and for $k \ge 0$, $j = 1,2,\ldots,6-k$. In other words, $P[X-Y=0] = \frac{1}{6}$, and we reduce this amount by $\frac{1}{36}$ for each step we take away from 0, either decreasing by one, or increasing by one.

#12 (C)

The outcome possibilities are, by definition, 11,12,21,22, while the same holds for the players' bets.

a Since the sum of the fingers shown can only be 2,3, or 4, these, their opposites $(-2, -3, -4)$, and 0, are the possible values for X

Consider the outcomes including the bets, and the corresponding values for X (each of the 16 lines has the same probability):

- 11 - 11 $X = 0$
- 11 - 12 $X = 2$
- 11 - 21 $X = -2$
- 11 - 22 $X = 0$
- 12 - 11 $X = -3$
- 12 - 12 $X = 0$
- 12 - 21 $X = 0$
- 12 - 22 $X = 3$
- 21 - 11 $X = 3$
- 21 - 12 $X = 0$
- 21 - 21 $X = 0$
- 21 - 22 $X = -3$
- 22 - 11 $X = 0$
- 22 - 12 $X = -4$
- 22 -21 $X = 4$
- 22 - 22 $X = 0$

A much faster way of computing probabilities would be to write the outcomes a different form: each player can throw the two numbers equally likely, and they can guess the right number equally likely. Hence we can count the outcomes as (R=right guess, W=wrong guess)
ij - RR $X = 0$
ij - RW $X = i + j$
ij - WR $X = -i - j$
ij - WW $X = 0$

Each line has probability $\frac{1}{4}$. Since $i+j=2$ or 4, with probability $\frac{1}{4}$, $i+j=3$ with probability $\frac{1}{2}$, we have that

$$P[X=0]=\frac{1}{2}$$

$$P[X=2]=P[X=-2]=P[X=4]=P[X=-4]=\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$$

$$P[X=3]=P[X=-3]=\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$$

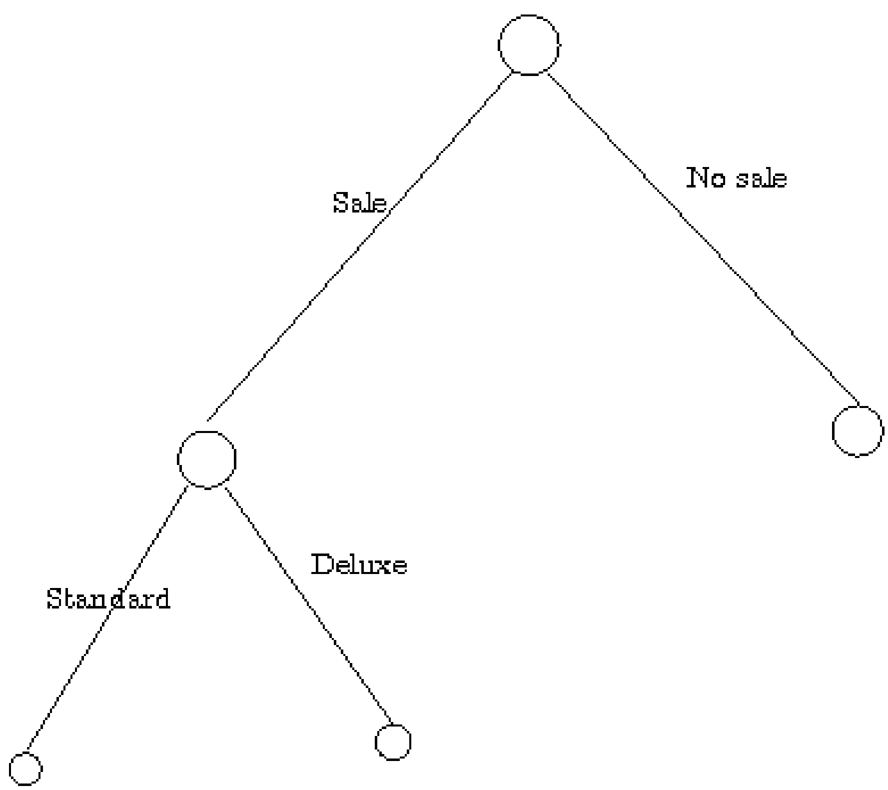
(it adds up to 1, ad it should: $\frac{1}{2}+4\cdot\frac{1}{16}+2\cdot\frac{1}{8}=1$)

b We are now ruling out results like 12 - 21, etc. That leaves only 4 possible outcomes (11 - 11, 12 - 12, 21 - 21, 22 -22), and either both guess right or both guess wrong, so the only possible outcome is $X=0$.

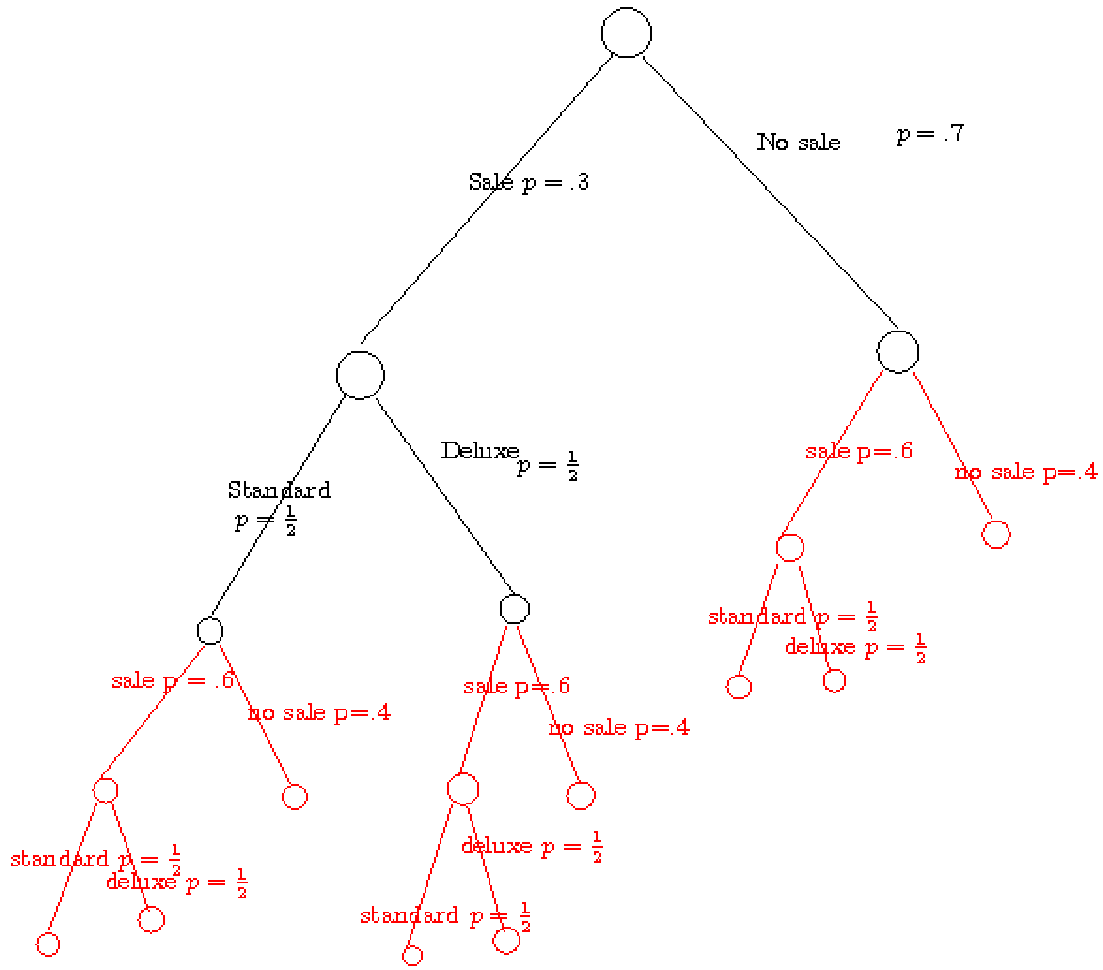
#13 (C)

We have six possible outcomes - three for each call. You can think of a tree for each call

For each call, the tree would look like this:



Combining the two sapes, we can solve the problem by calculating the probabilities for all "leaves (the end nodes) of the complete tree:



Alternatively, we can compute the outcome of each call, and add the results, relying on the independence of the calls. This would go like this:

Using conditional probability calculations on each tree, we have that if A means sale of standard, and B sale of deluxe,

$$P[A] = P[B] = P[A\,|\text{sale}]\,P[\text{sale}] = .5 \cdot .3$$

for the first call, and

$$P[A] = P[B] = P[A\,|\text{sale}]\,P[\text{sale}] = .5 \cdot .6$$

for the second. If X_1, and X_2 are the two dollar values, we have

$$P[X_1 = 500] = P[X_1 = 1000] = .15$$

$$P[X_2 = 500] = P[X_2 = 1000] = .3$$

Now, $X = X_1 + X_2$, the sum of two independent variables. Without using the convolution formula introduced in point **3** of problem (but we could!), we have

$$P[X = 0] = P[X_1 = X_2 = 0] = P[X_1 = 0]\, P[X_2 = 0] = .7 \cdot .4 = .28$$

$$P[X = 500] = P[X_1 = 500, X_2 = 0] + P[X_1 = 0, X_2 = 500] = .15 \cdot .4 + .7 \cdot .3 = .27$$

$$P[X = 1000] = P[X_1 = 1000, X_2 = 0] + P[X_1 = 500, X_2 = 500] + P[X_1 = 0, X_2 = 1000] =$$

$$= .15 \cdot .4 + .15 \cdot .3 + .7 \cdot .3 = .315$$

$$P[X = 1500] = P[X_1 = 1000, X_2 = 500] + P[X_1 = 500, X_2 = 1000] = .15 * .3 + .15 * .3 = .09$$

$$P[X = 2000] = P[X_1 = 1000, X_2 = 1000] = .15 * .3 = 0.045$$

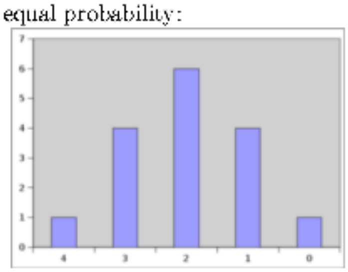
Of course, there are other ways to define the events $\{X = x\}$, and hence of computing the pmf for X.

#18

The "basic" events are $HHHH, HHHT, HHTH, \ldots, TTTT$. There are $2^4 = 16$ of them, each has probability $\frac{1}{2^4} = \frac{1}{16}$. The number of them that involve 4 **H** is 1, **3 H** is 4, **2 H** is $\binom{4}{2} = 6$, 1 **H** is 4, and no **H** is 1. So we have, calling X the number of heads,

$$P[X = j] = \binom{4}{j} \frac{1}{2^4}$$

You will notice the typical "tent" or "peaked" and symmetric shape of this distribution: it characterizes the *binomial distribution* when we are dealing with 0-1 variables with equal probability:



Now, we are asked for the pmf of $X-2$: clearly $P[X - 2 = k] = P[X = k + 2]$, and so

$$P[X - 2 = k] = \binom{4}{k+2} \frac{1}{2^4}$$

for $k = -2, -1, 0, 1, 2$. The corresponding histogram is exactly the same, except it is *shifted horizontally* (to the left) by -2.

#20

The various possibilities are, as so often, well represented by a tree graph. With probability $\frac{18}{38}$, $X = 1$ and the play ends. With probability $\frac{20}{38}$, we make *two* bets on red. These produce the outcomes

1. $X = -3$ if no red comes up at all - conditional probability $\left(\frac{20}{38}\right)^2$
2. $X = -1$ if one red comes up - conditional probability $2 \cdot \frac{20}{38} \cdot \frac{18}{38}$
3. $X = 1$ if two reds come up - conditional probability $\left(\frac{18}{38}\right)^2$

Overall,

$$P[X = i] = \begin{cases} \left(\frac{20}{38}\right)^3 & i = -3 \\ 2 \cdot \left(\frac{20}{38}\right)^2 \cdot \frac{18}{38} & i = -1 \\ \frac{18}{38} + \frac{20}{38} \cdot \left(\frac{18}{38}\right)^2 & i = 1 \end{cases}$$

a From the table, $P[X > 0] = P[X = 1] = \frac{18}{38} + \frac{20}{38} \cdot \left(\frac{18}{38}\right)^2 = .5918$

b This is the wrong place to ask this question. You should wait for the result in point c! The answer in the answer book ("no, because when the gambler wins she wins \$1, but when she loses she loses \$1 or \$3") is meaningless, since, as an example, if she had 99% probability of winning, and .0001% probability of losing \$3, I would suggest that she should adopt this strategy!

c From the table, we can compute

$$EX = \left[\frac{18}{38} + \frac{20}{38} \cdot \left(\frac{18}{38}\right)^2\right] \cdot 1 + 2 \cdot \frac{18}{38} \cdot \left(\frac{20}{38}\right)^2 \cdot (-1) + \left(\frac{20}{38}\right)^3 \cdot (-3) =$$
$$= -.108$$

so that, even she is likely to win more than half the time, in the long run she can expect to lose money, since every time she loses the loss is, on average, big enough to offset the wins.

#22

You can use this problem as an excuse to evaluate (roughly!) the "home field advantage", in baseball and basketball playoffs. We are playing a "best of $i+i-1$" series. Consider team A and count its number of wins, X. The series ends when i 1s or i 0s have appeared over $2i-1$ trials. Even if it's not part of the problem, let's look at the probability of team A winning the series. There is no need to do calculation by stopping as soon as the i 1s or 0s have appeared: we can imagine that $2i-1$ games are played anyway (this is what is usually done in

Davis Cup and Federation Cup Tennis series). Hence the probability that team A has "i or more" wins is

$$P[X \geq i] = \sum_{j=i}^{2i-1} \binom{2i-1}{j} p^j (1-p)^{2i-j-1}$$

Going to the problem at hand, the number of games played is determined by the position of the ith win for either team. Plainly, we can look at the probability that the series lasts $i, i+1, \ldots, 2i-1$ games when A wins, and obtain the result for the other team, simply by interchanging p with $1-p$. Since the result is, in this sense, symmetric, and the result is that there are i games with probability one whenever $p=0$ or $p=1$, we can guess that, indeed, the highest expected number of games should occur when $p=\frac{1}{2}$.

To find a general formula, we note that if we play $i+k$ games, we need to count the ways the k wins by the other team can be intermingled in $i+k$ games - $\binom{i+k}{k}$. However, we need to subtract the ways where the $(i+k)$th win is not A's. These are exactly the number of series where A wins in $i, i+1, \ldots, i+k-1$ games. We thus have a recursive solution:

- There is $\binom{i}{0} = 1$ way to win in i games
- There are $\binom{i+1}{1} - 1 = i$ ways to win in $i+1$ games
- There are $\binom{i+2}{2} - \binom{i+1}{1} - \binom{i}{0} = \frac{(i+2)(i+1)}{2} - (i+1) - 1$ ways to win in $i+2$ games
- ...
- There are $\binom{i+i-1}{i-1} - \binom{i+i-2}{i-2} - \ldots - \binom{i}{0}$ ways to win in $2i-1$ games

Applying, if N is the number of games played,

$$P[N = i+k \,|X \geq i] = \left(\binom{i+k}{k} - \sum_{j=0}^{k-1} \binom{i+j}{j} \right) p^i (1-p)^k \,; k = 0, 1, \ldots, i-1$$

$$EN = \sum_{k=0}^{i-1} (i+k) \left(\binom{i+k}{k} - \sum_{j=0}^{k-1} \binom{i+j}{j} \right) \left(p^i (1-p)^k + p^k (1-p)^i \right)$$

Finally, we can also note that the highest terms ($k = i-1$) have a contribution from the last parenthesis of the form

$$p^i (1-p)^{i-1} + p^{i-1} (1-p)^i = p^{i-1} (1-p)^{i-1} (p+1-p) = p^{i-1} (1-p)^{i-1} = -p^{2i-2} + p^{i-1}$$

We see that we always get a polynomial of even degree $2i-2$, that, because of the symmetry of the problem, has to be symmetric around $p = \frac{1}{2}$ (interchanging p with $1-p$ the problem simply interchanges A with the other team). This means that $p = \frac{1}{2}$ *has* to be an extremum for the polynomial, and, still by symmetry, this has to be a maximum: it is at least intuitive that no other zero of the derivative can occur between 0 and 1, and that the function is increasing when p grows from 0, and decreasing as it grows towards 1.

In the two special cases mentioned

a $i = 2$: $EN = 2\left(p^2 + (1-p)^2\right) + 3\,(2p\,(1-p)) = 2 + 2p\,(1-p) = 2 + 2p - 2p^2$. This has a parabola graph, concave down, and the vertex (the maximum) is midway between two equal levels, e.g. midway between 0 and 1 (both give $EN = N = 2$), as suspected. Yes, you can also derive and find the result by zeroing the derivative $2 - 4p$, even though it looks like overkill.

b $i = 3$:

$$EN = 3\left(p^3 + (1-p)^3\right) + 4\left(3p^3\,(1-p) + 3p\,(1-p)^3\right) + 5 \cdot 6p^2\,(1-p)^2 =$$

$$= 6p^4 - 12p^3 + 3p^2 + 3p + 3$$

which, again, is equal to 3 for $p = 0$, and $p = 1$, and, by symmetry, will have a maximum in the middle. Using derivatives gives a similar result, except you have to solve a 3rd degree equation

$$24p^3 - 36p^2 + 6p + 3 = 0$$

find the other zeros (that is easy, since you can check directly that $p = 0.5$ is a root), and check that the maximum does correspond to $p = 0.5$. Dividing by $p - \frac{1}{2}$ (if you didn't know, you can apply the standard long division algorithm to divide two polynomials), we obtain

$$24p^2 - 24p - 6$$

and equating to zero, the remaining critical points are the solutions to

$$4p^2 - 4p - 1 = 0$$

or $p = \frac{1}{2}\left(1 \pm \sqrt{2}\right)$. Since neither is in the range $[0, 1]$, $\frac{1}{2}$ is the only extremum we are interested, and by symmetry, since, formally, $EN \to +\infty$ when $|p| \to \infty$, the two "outer" critical points have to be minima, and the middle critical point has to be a maximum. OK, you can also get the result by checking the sign of the second derivative.

#23 (C)

If you but $\$x$ of the commodity today, you will have $\frac{x}{2}$ ounces, and they will be worth either $\$\frac{x}{2}$ or $\$4 \cdot \frac{x}{2}$ in a week.

a Your expected wealth (in the commodity) will be $E\$ = \frac{1}{2}\left(\frac{x}{2} + 2x\right) = x + \frac{x}{4}$. This is linear in x, so the maximum is at the highest possible value for x, i.e. \$1000. The expected wealth is $\$\,(1000 + 250) = \$1250$

b If you buy today $\$x$ of the commodity, and $\$\,(1000 - x)$ in a week, you will get $\frac{x}{2}$ ounces today, and either $1000 - x$ or $\frac{1000-x}{4}$ ounces in a week. The expected number of ounces will be

$$\frac{x}{2} + \frac{1}{2}\left(1000 - x + \frac{1000 - x}{4}\right) = 625 - \frac{x}{8}$$

which is again linear, so that the maximum occurs at the lowest value of x, i.e. $x = 0$.

#25

This is, essentially, a tedious exercise in listing:

we need to compute the probabilities for each combination in the second table, using the probabilities:

Dial 1: $P\,[\text{Bar}] = \frac{1}{20}, P\,[\text{Bell}] = \frac{1}{10}, P\,[\text{Plum}] = \frac{1}{5}, P\,[\text{Lemon}] = \frac{3}{20}, P\,[\text{Orange}] = \frac{3}{20}, P\,[\text{Cherry}] = \frac{7}{20}$

Dial 2: $P\,[\text{Bar}] = \frac{3}{20}, P\,[\text{Bell}] = \frac{1}{10}, P\,[\text{Plum}] = \frac{1}{20}, P\,[\text{Lemon}] = 0, P\,[\text{Orange}] = \frac{7}{20}, P\,[\text{Cherry}] = \frac{7}{20}$

Dial 3: $P\,[\text{Bar}] = \frac{1}{20}, P\,[\text{Bell}] = \frac{3}{20}, P\,[\text{Plum}] = \frac{3}{10}, P\,[\text{Lemon}] = \frac{1}{5}, P\,[\text{Orange}] = \frac{3}{10}, P\,[\text{Cherry}] = 0$

The various combinations have probabilities

1. Bar-Bar-Bar (60): $\frac{1}{20} \cdot \frac{3}{20} \cdot \frac{1}{20} = \frac{3}{8000} = .000375$
2. Bell-Bell-Bell (20): $\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{3}{20} = \frac{3}{2000} = .0015$
3. Bell-Bell-Bar (18): $\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{20} = \frac{1}{2000} = .0005$
4. Plum-Plum-Plum (14): $\frac{1}{5} \cdot \frac{1}{20} \cdot \frac{3}{10} = \frac{3}{1000} = .003$
5. Orange-Orange-Orange (10): $\frac{3}{20} \cdot \frac{7}{20} \cdot \frac{3}{10} = \frac{63}{4000} = .1575$
6. Orange-Orange-Bar (8): $\frac{3}{20} \cdot \frac{7}{20} \cdot \frac{1}{20} = \frac{21}{8000} = .002625$
7. Orange-Cherry-Anything (2): $\frac{3}{20} \cdot \frac{7}{20} \cdot 1 = \frac{21}{400} = .0525$
8. Cherry-No Cherry-Anything (0): $\frac{7}{20} \cdot \frac{13}{20} \cdot 1 = \frac{91}{400} = .2275$
9. Anything else (-1):

$$1 - \frac{3 + 12 + 4 + 24 + 126 + 21 + 420 + 1820}{8000} = 1 - \frac{2414}{8000} = .69625$$

It is already evident that the payoff schedule is not really rational, and also extremely unfair (roulette, blackjack, craps, and other casino table games have much better odds for the players, even as they still favor the house). The expected payoff for a player is, using the data above

$$EX = 60\cdot\frac{3}{8000}+20\cdot\frac{12}{8000}+18\cdot\frac{4}{8000}+14\cdot\frac{24}{8000}+10\cdot\frac{126}{8000}+8\cdot\frac{21}{8000}+2\cdot\frac{420}{8000}-1\cdot\frac{5586}{8000} =$$

$$= -\frac{2490}{8000} = -0.31125$$

Theoretical Exercises From Chapter 4 (Part 1)

#3

The (cumulative) distribution function for a random variable X is defined as

$$F_X(x) = P[X \leqslant x]$$

(this is just traditional, and results in a function that is *continuous from the right, with left-hand limits* – but we could, just as well, have chosen $P[X < x]$, except that this tradition is now consolidated).

Since we have that $1 - F_x(x) = P[X > x]$, and $P[X = x] = P[\{X \leqslant x\} \backslash \{X < x\}] = F_X(x) - \lim_{y \uparrow x} F_X(y)$,

$$P[X \geqslant x] = 1 - F_X(x) + F_X(x) - \lim_{y \uparrow x} F_X(y) = 1 - \lim_{y \uparrow x} F_X(y)$$

#4

The (cumulative) distribution function of e^X will be

$$P\left[e^X \leqslant x\right] = P[X \leqslant \log(x)] = F(\log(x))$$

Remark 1. Here we are following the notation used by the book, where "log" stands for *natural logarithm*. This is also the convention used in practically all mathematical and physics texts above the algebra/introductory calculus level, since natural logarithms are the only ones that anybody would consider using. Recall that decimal logarithms are not practical as *functions*: they are still in use in situations like the evaluation of the pH of a solution, or the sound level in decibels. These are all static numbers, and since they are converting from *scientific notation*, decimal logs are convenient.

#5

Again, we only need to apply the definition:

$$P[\alpha X + \beta \leqslant x] = P\left[X \leqslant \frac{x - \beta}{\alpha}\right] = F\left(\frac{x - \beta}{\alpha}\right)$$

#6

We have

$$EN = \sum_{i=1}^{\infty} i\, P[N = i] = \sum_i i(P[N \geq i - 1] - P[N \geqslant i])$$

Since $\sum_i i\, P[N \geqslant i - 1] = \sum_i (i + 1) P[N \geqslant i]$, the conclusion follows immediately.

Remark 2. We changed the summation index in the last formula. This proof does not follow the hint from the book. Try that method, as "inverting summation order" is a useful trick in many situations.

#7

The hint from the book works immediately, but let's try another route:

$$EN^2 - EN = \sum_{i=1}^{\infty} i^2 P[N=i] - \sum_{i=1}^{\infty} i P[N=i] =$$

$$= \sum_{i=1}^{\infty} i^2 (P[N > i-1] - P[N > i]) - \sum_{i=1}^{\infty} i (P[N > i-1] - P[N > i]) =$$

$$= \sum_{i=0}^{\infty} (i+1)^2 P[N > i] - \sum_{i=0}^{\infty} i^2 P[N > i] - \sum_{i=0}^{\infty} (i+1) P[N > i] + \sum i P[N > i] =$$

$$= \sum_{i=0}^{\infty} (2i + 1 - i - 1 + i) P[N > i] = 2 \sum_{i=0}^{\infty} P[N > i]$$

#8 (not suggested, but, oh, well...)

We will have that

$$E\left[c^X\right] = pc + (1-p)c^{-1}$$

$$pc + (1-p)\, c^{-1} = 1 \Longleftrightarrow pc^2 - c + 1 - p = 0$$

$$c = \frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p}$$

This has always a real solution since $p(1-p) \leqslant \frac{1}{4}$, with the maximum occurring at $p = \frac{1}{2}$. Also,

$$\sqrt{1 - 4p + 4p^2} = \sqrt{(1-2p)^2} = |1 - 2p|$$

We see that

- for $p \leqslant \frac{1}{2}$, the "$+$" solution is equal to $\frac{1+1-2p}{2p} = \frac{1-p}{p}$, while the "$-$" solution is $\frac{1-1+2p}{2p} = 1$
- for $p \geqslant \frac{1}{2}$, the "$+$" solution is equal to $\frac{1+2p-1}{2p} = 1$, while the "$-$" solution is $\frac{1-2p+1}{2p} = \frac{1-p}{p}$

Thus we have a continuous solution equal to $\frac{1-p}{p}$ for $0 < p \leqslant 1$, and another continuous solution, equal to 1, for all p.

The following graph shows both solutions: the “+” is in blue, the “ − ” in green:

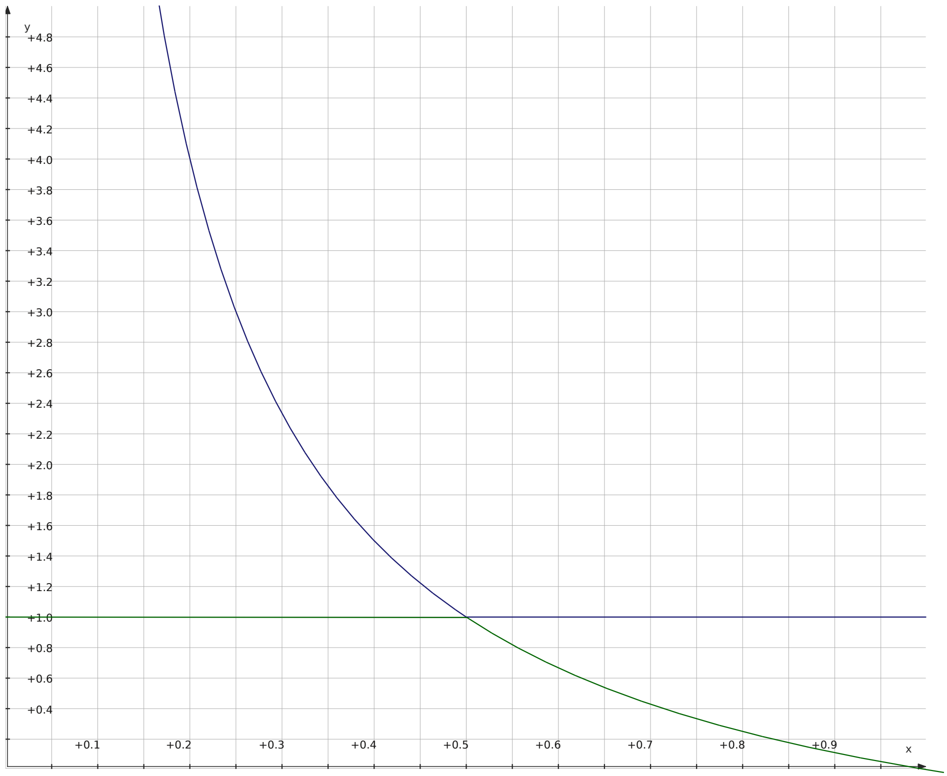


Figure 1.

#9

This is a commonly used fact. Indeed, in general,

$$E[\alpha X+\beta]=\alpha EX+\beta$$
$$\mathrm{Var}[\alpha X+\beta]=E\Big[(\alpha X+\beta-\alpha EX-\beta)^2\Big]=E\Big[\alpha^2(X-EX)^2\Big]=\alpha^2\mathrm{Var}[X]$$

Hence, having set $\mu=EX,\sigma^2=\mathrm{Var}[X]$, we have from $Y=\frac{1}{\sigma}X-\frac{\mu}{\sigma}$,

$$EY=\frac{1}{\sigma}EX-\frac{\mu}{\sigma}=0$$
$$\mathrm{Var}[Y]=\frac{1}{\sigma^2}\mathrm{Var}[X]=1$$