Math 394B Summer 2010

Solutions to Homework Assignment

Problems from Chapter 3

#2

Let A be the event that the *first* die turns a 6, and B_i the event that the sum of the dice is i. Now, $P[A \cap B_i] = P[(6, i-6)]$, and this is the same as the probability of two dice coming up with two specified points - $\frac{1}{36}$, for $i = 7, 8, \ldots, 12$ (if $i \leq 6$, the intersection is clearly empty). By definition of conditional probability,

$$P[A|B_i] = \frac{P[A \cap B_i]}{P[B_i]} = \frac{1}{36} \cdot \frac{1}{P[B_i]}$$

for $i = 7, 8, \ldots, 12$. Since,

$$P[B_7] = \frac{6}{36}, P[B_8] = \frac{5}{36}, P[B_9] = \frac{4}{36}, P[B_{10}] = \frac{3}{36}, P[B_{11}] = \frac{2}{36}, P[B_{12}] = \frac{1}{36}$$

(i.e., $P[B_i] = \frac{13-i}{36}$), we get

$$P[A|B_i] = \frac{1}{13-i}$$

#7 (C)

This can be handled with a very small sample space: two children, either can be a boy or a girl, hence the possible outcomes are BB, BG, GB, GG. The event "one of them (the king) is a boy" is $B = \{BB, BG, GB\}$. Conditioning on B, if A is the event that one of the children is a girl, $A \cap B = \{BG, GB\}$, and, if we assume that all four outcomes were a priori of equal probability, knowing that one of the children is a boy (the king), gives a conditional probability for a sister as

$$\frac{\frac{2}{4}}{\frac{3}{4}} = \frac{2}{3}$$

The same result is obtained if we start by considering B (the event we already know happened) as the sample space, with each of its three points having the same probability.

#8 (C)

There is a big difference from the previous problem. Here, we know the gender of a *specific* child (the first born), not of *one of the children*. The known event is now $B = \{GB, GG\}$, with probability $\frac{1}{2}$, and the event that both are girls is a subset of B, $A = \{GG\}$, and has probability $\frac{1}{4}$. Hence, the conditional probability $P[A|B] = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$.

15

This is an example of inductive reasoning, employing Bayes' Rule. Let S be the event "a randomly chosen woman smokes", and E the event that a randomly chosen woman has an ectopic pregnancy.

We are told that ($S^c = \Omega \setminus S$ is the event that a randomly chosen woman is a non smoker)

- $P[S] = .32$
- $P[E|S] = 2 \cdot P[E|S^c]$

and we are requested to evaluate $P[S|E]$.

Applying Bayes' rule, we have

$$P[S|E] = \frac{P[E|S]\,P[S]}{P[E|S]\,P[S] + P[E|S^c]\,P[S^c]} \tag{1}$$

We know that $P[S^c] = 1 - P[S] = .68$, and that $P[E|S] = 2P[E|S^c]$. Substituting the latter in (1), and dividing both numerator and denominator by $P[E|S^c]$, we have

$$P[S|E] = \frac{2 \cdot .32}{2 \cdot .32 + .68} = .48485$$

#16

Let C be the event of a Cesarean section. We have that $P[C] = .15$. Let S be the event that the child survives birth. We are told that $P[S] = .98$, and that $P[S|C] = .96$. We are asked for $P[S|C^c]$.

Clearly,

$$P[S] = P[S \cap C] + P[S \cap C^c] = P[S|C]\,P[C] + P[S|C^c]\,P[C^c]$$

and, plugging in our numbers, this means

$$.98 = .96 \cdot .15 + P[S|C^c] \cdot .85$$

or

$$P[S|C^c] = \frac{.98 - .96 \cdot .15}{.85} = .98353$$

21

Consider the events H (husband earns more than $25,000), and W (wife earns more than $25,000). The table means

$$P[H \cap W] = \frac{54}{500}$$

$$P[H \cap W^c] = \frac{198}{500}$$

$$P[H^c \cap W] = \frac{36}{500}$$

$$P[H^c \cap W^c] = \frac{212}{500}$$

$$P[H] = \frac{198+54}{500} = \frac{252}{500}$$

$$P[W] = \frac{36+54}{500} = \frac{90}{500}$$

It is now easy to answer:

a $P[H^c] = 1 - \frac{252}{500} = .496$

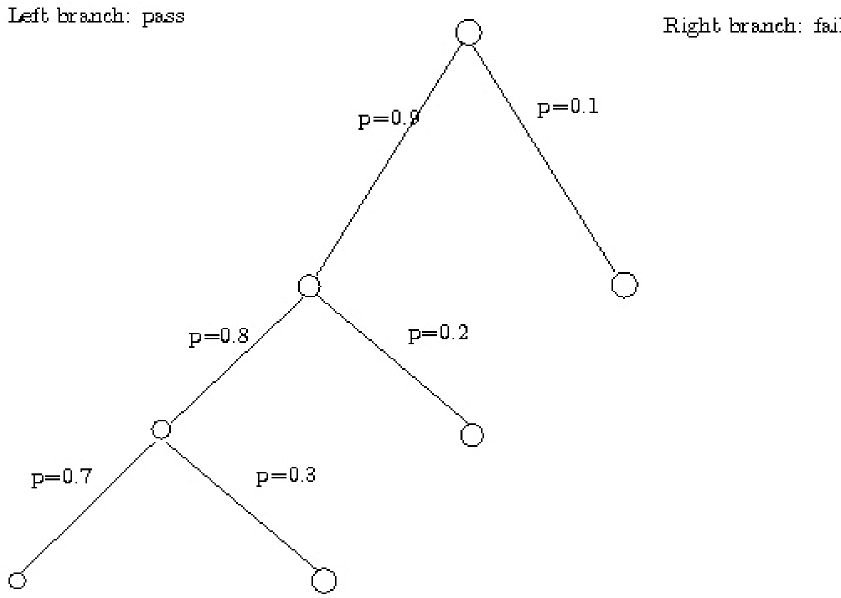
b $P[W\,|H] = \frac{P[H \cap W]}{P[H]} = \frac{54}{252} = .21429$

c $P[W\,|H^c] = \frac{P[W \cap H^c]}{P[H^c]} = \frac{36}{248} = .145$

Note that the last two answers do not have any specific relationship with each other: they refer to two different conditions, and hence result in unrelated probabilities. They definitely do not add up to one!

12 (C)

It is useful to represent the situation as a tree: each node represents an exam, with two branches departing, depending on whether the graduate passes or fails. We can attach to each branch the probability of taking it, *conditioned on the path above it:*



(We'll see many other examples where such a picture can be helpful). Now, if we call the events of passing exams 1, 2, and 3 E_1, E_2, E_3, we have

$$P[E_1 \cap E_2 \cap E_3] = P[E_3 | E_1 \cap E_2] \, P[E_1 \cap E_2] =$$

$$P[E_3 | E_1 \cap E_2] \, P[E_2 | E_1] \, P[E_1]$$

In other words, the probability of reaching the lowest rightmost "leaf", is given by the product of the conditional probabilities of the branches leading to that leaf.

a We just found that the probability of passing all three exams is $.9 \cdot .8 \cdot .7 = .504$

b From the picture, we see that

$$P[E_1^c] = .1$$

$$P[E_2^c] = P[E_1] \, P[E_2^c | E_1] = .9 \cdot .2 = .18$$

$$P[E_3^c] = P[E_1] \, P[E_2 | E_1] \, P[E_3^c | E_1 \cap E_2] = .9 \cdot .8 \cdot .3 = .216$$

Since she cannot fail more than one exam, these three events are disjoint, and their union represents the event we are conditioning to. The event of failing the second exam is a subset of this, so the intersection with it is, again, E_2^c. All in all, the conditional probability we are looking for is

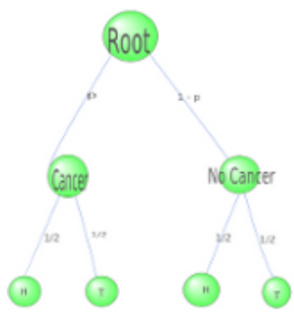
$$\frac{.18}{.1+.18+.216} = .3629$$

27(C)

In a way, the "correct" answer depends on what you are asking precisely. As stated, the question seems to be "given a randomly chosen car, how many workers rode on it?", and hence the correct procedure is *2.* Procedure *1.* answers the question "given a randomly chosen worker, with how many co-workers did he/she share a ride?" - which is a different question, and has, in general, a different answer (big carpools enhance the probability of picking a worker who has many co-riders, but that does not necessarily mean that many cars are pooled).

#31 (C)

We can proceed in a number of ways. For example, we can use a tree diagram, like the following one:



(for typographical reason the probability called α in the problem is denoed by$p$ in this diagram).

Alternatively, one can contruct a picture of a sample space, with four events, C, C^c, H, T that intersect to form four atoms, which we can call CH, CT, C^cH, C^cT. We have $P[C] = \alpha = 1 - P[C^c]$, and $P[H] = P[T] = \frac{1}{2}$. Finally, C is independent of H and T, as is C^c. In fact, it is more elegant to consider two independent random variables, each taking on two values – essentially the indicator functions, respectively, of C and H.

Given independence,

$$P[CH] = \frac{\alpha}{2}, P[CT] = \frac{\alpha}{2}, P[C^cH] = \frac{1-\alpha}{2}, P[C^cT] = \frac{1-\alpha}{2}$$

Given the rule, the event "the doctor does not call", call it N, is the complement of the only case in which he would call, which is C^cH. Hence, $P[N] = 1 - P[C^cH] = 1 - \frac{1-\alpha}{2} = \frac{2-1+\alpha}{2} = \frac{1+\alpha}{2}$ (it is also, of course, the sum of the probabilities of the other three atoms).

We are looking for $\beta = P[C|N] = \frac{P[C \cap N]}{P[N]}$. Now, since $N = CH \cup CT \cup C^cT$, it is clear that $C \subset N$, so that $P[C \cap N] = P[C] = \alpha$. Summing it all up,

$$\beta = P[C|N] = \frac{P[C \cap N]}{P[N]} = \frac{P[C]}{P[N]} = \frac{\alpha}{\frac{1+\alpha}{2}} = \frac{2\alpha}{1+\alpha}$$

To compare α with β, we calculate

$$\alpha - \beta = \alpha - \frac{2\alpha}{1+\alpha} = \frac{\alpha + \alpha^2 - 2\alpha}{1+\alpha} = \frac{\alpha^2 - \alpha}{1+\alpha} = \frac{\alpha(\alpha-1)}{1+\alpha} < 0$$

since $0 < \alpha < 1$. While a call can only bring good news, not receiving a call, quite intuitively, means that the probability of bad news has increased.

#48

We are in the region of the questions of type "three prisoners", "Monte Hall", etc. You would think that the probability should be $\frac{1}{2}$, but look closer...

We are going to solve the problem in the "automatic way", i.e., by applying Bayes' Rule. A classroom note will be posted illustrating more "hands-on" solutions that may (or may not) help figure out the mechanism of these problems.

We have that two events are SS and SG, describing the two cabinets. We found S, and ask for the conditional probability of SS, given S. Now, $P[S|SS] = 1$, $P[S|SG] = \frac{1}{2}$. Also, we assume that the choice of cabinet was "random", i.e. both have probability $\frac{1}{2}$ (this assumption is crucial in determining the correct answer - see the classroom note).

$$P[SS|S] = \frac{P[S|SS]\,P[SS]}{P[S]}$$

and $P[S] = P[S|SS]\,P[SS] + P[S|G]\,P[G] = 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$. Substituting

$$P[SS|S] = \frac{1 \cdot \frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

#49

This is an example of application of Bayes' Rule, as well as an illustration of the power of "prior beliefs". Consider the event C, that a random patient has prostate cancer. We have, in the first version, that, according to the physician, $P[C] = 0.7$. Let T be the event that the test was positive (i.e., showed an elevated PSA level). We are told that $P[T|C] = 0.268$, and $P[T|C^c] = 0.135$.

We apply Bayes' Rule to find the "reversed" conditional probabilities, which are the ones we are asked for:

$$P[C|T] = \frac{P[T|C]P[C]}{P[T]} = \frac{P[T|C]P[C]}{P[T|C]P[C] + P[T|C^c]P[C^c]} =$$

$$= \frac{0.268 \cdot 0.7}{0.268 \cdot 0.7 + 0.135 \cdot 0.3} \approx 0.82$$

$$P[C|T^c] = \frac{P[T^c|C]P[C]}{P[T^c]} = \frac{P[T^c|C]P[C]}{P[T^c|C]P[C] + P[T^c|C^c]P[C^c]} =$$

$$= \frac{0.732 \cdot 0.7}{0.732 \cdot 0.7 + 0.865 \cdot 0.3} \approx 0.66$$

Reversing the "prior probabilities" of the patient having cancer, we find

$$P[C|T] = \frac{0.268 \cdot 0.3}{0.268 \cdot 0.3 + 0.135 \cdot 0.7} \approx 0.46 \qquad P[C|T^c] = \frac{0.732 \cdot 0.3}{0.732 \cdot 0.3 + 0.865 \cdot 0.7} \approx 0.27$$

52 (C)

Let A be the event the student is accepted: $P[A] = .6$. We are also given a table for all the conditional probabilities for acceptance and rejection, for each day of the week. Actually, we have to be careful in reading the table. What it is saying is that we have disjoint events M, T, W, Th, F, nm, indicating the day of the week in which the mail might arrive (nm meaning no mail arrives in the week), forming a partition of the sample space, and two disjoint events A, R (accepted/rejected) that also form a partition of the sample space. We have

$$P[M|A] + P[T|A] + P[W|A] + P[Th|A] + P[F|A] + P[nm|A] = 1$$

and similarly when we condition on R. For example, this means that

$$M^c = T \cup W \cup Th \cup F \cup nm$$

etc. Also, we find that $P[nm|A] = 1 - .85 = .15, P[nm|R] = 1 - .6 = .4$, and $P[nm] = P[nm|A]P[A] + P[nm|R]P[R] = .15 \cdot .6 + .4 \cdot .4 = .25$

Applying "total probabilities", Bayes' Rule, etc. we have

a $P[M] = P[M|A]P[A] + P[M|R]P[R] = .15 \cdot .6 + .05 \cdot .4 = .11$

b $P[T|M^c] = \frac{P[T \cap M^c]}{P[M^c]}$, but $T \subseteq M^c$, so the fraction is

$$\frac{P[T]}{P[M^c]} = \frac{P[T|A]P[A] + P[T|R]P[R]}{P[M^c]} = \frac{.2 \cdot .6 + .1 \cdot .4}{1 - .11} = .17978$$

c $P[A|M^c \cap T^c \cap W^c] = \frac{P[M^c \cap T^c \cap W^c|A]P[A]}{P[M^c \cap T^c \cap W^c]}$ Now, $M^c \cap T^c \cap W^c = Th \cup F \cup nm$, so that

$$[M^c \cap T^c \cap W^c|A]P[A] = P[Th|A]P[A] + P[F|A]P[A] + P[nm|A]P[A]$$

$$P[M^c \cap T^c \cap W^c] = P[Th|A]P[A] + P[F|A]P[A] + P[nm|A]P[A] +$$

$$+ P[Th|R]P[R] + P[F|R]P[R] + P[nm|R]P[R]$$

$$P[A|M^c \cap T^c \cap W^c] = \frac{.6 \cdot (.15 + .1 + .15)}{6 \cdot (.15 + .1 + .15) + .4 \cdot (.15 + .2 + .4)} = .4444$$

d This a straight application of Bayes' Rule:

$$P[A|Th] = \frac{P[Th|A]\,P[A]}{P[Th]} = \frac{.15 \cdot .6}{.15 \cdot .6 + .5 \cdot .4} = .6$$

e Again, apply Bayes' Rule, and find

$$P[A|nm] = \frac{P[nm|A]}{P[nm]}P[A]$$

From the table, we get that

$$= \frac{.15 \cdot .6}{.15 \cdot .6 + .4 \cdot .4} = .36$$

Theoretical Exercises

#1

We have

$$P\left[A\bigcap B\left|A\bigcup B\right.\right]=\frac{P\left[(A\bigcap B)\bigcap(A\bigcup B)\right]}{P\left[A\bigcup B\right]}=\frac{P\left[(A\bigcap B)\bigcap B\right]}{P\left[A\bigcup B\right]}=\frac{P\left[A\bigcap B\right]}{P\left[A\bigcup B\right]}$$

and since $P[A]\le P[A\bigcup B]$, we have that

$$P\left[A\bigcap B\left|A\bigcup B\right.\right]\le P\left[\left|A\bigcap B\ A\right.\right]=\frac{P\left[A\bigcap B\right]}{P[A]}$$

#5

The assumption is that

$$P\left[E\left|F\right.\right]=\frac{P\left[E\bigcap F\right]}{P[F]}\le P[E]$$

that is

$$P\left[E\bigcap F\right]\le P[E]\,P[F] \tag{1}$$

(the difference between the two sides is sometimes called the "correlation" between the two events, and, in this case, one speaks of "negative correlation")

(a) (1) is symmetric in the two events (just as the concept of "independence"), so the statement is true. The same argument applies with the inequality reversed)"positive correlation")

(b) A simple counterexample is given by the case when $G\subset E, F\bigcap E=\emptyset$:

$$P\left[E\bigcap F\right]=P\left[G\bigcap F\right]=0\le P[E]\,P[F]\ \text{ and }\ \le P[G]\,P[F]$$

while $P[EG]=P[G]\ge P[E]\,P[G]$ (all inequalities will be strict, provided none of the sets has probability 0 or 1).

#25

We are comparing $\frac{P[E\bigcap F]}{P[F]}$ with $\frac{P[E\bigcap F\bigcap G]}{P[F\bigcap G]}\cdot\frac{P[F\bigcap G]}{P[F]}+\frac{P[E\bigcap F\bigcap G^c]}{P[F\bigcap G^c]}\cdot\frac{P[F\bigcap G^c]}{P[F]}$. Equality is obvious, since

$$E\bigcap F=\left(E\bigcap F\bigcap G\right)\bigcup\left(E\bigcap F\bigcap G^c\right)$$

with the right hand side being the union of disjoint sets.

Please, note the equality that this proves: it shows "how to condition a conditional statement", if we may use this awkward expression.