Math 394 B & C

Solutions to Homework Assignment

Problems from Chapter 3

#2

Let A be the event that the *first* die turns a 6, and B_i the event that the sum of the dice is i. Now, $P[A \cap B_i] = P[(6, i-6)]$, and this is the same as the probability of two dice coming up with two specified points - $\frac{1}{36}$, for $i = 7, 8, \ldots, 12$ (if $i \leq 6$, the intersection is clearly empty). By definition of conditional probability,

$$P[A|B_i] = \frac{P[A \cap B_i]}{P[B_i]} = \frac{1}{36} \cdot \frac{1}{P[B_i]}$$

for $i = 7, 8, \ldots, 12$. Since,

$$P[B_7] = \frac{6}{36}, P[B_8] = \frac{5}{36}, P[B_9] = \frac{4}{36}, P[B_{10}] = \frac{3}{36}, P[B_{11}] = \frac{2}{36}, P[B_{12}] = \frac{1}{36}$$

(i.e., $P[B_i] = \frac{13-i}{36}$), we get

$$P[A|B_i] = \frac{1}{13-i}$$

#7 (C)

This can be handled with a very small sample space: two children, either can be a boy or a girl, hence the possible outcomes are BB, BG, GB, GG. The event "one of them (the king) is a boy" is $B = \{BB, BG, GB\}$. Conditioning on B, if A is the event that one of the children is a girl, $A \cap B = \{BG, GB\}$, and, if we assume that all four outcomes were a priori of equal probability, knowing that one of the children is a boy (the king), gives a conditional probability for a sister as

$$\frac{\frac{2}{4}}{\frac{3}{4}} = \frac{2}{3}$$

The same result is obtained if we start by considering B (the event we already know happened) as the sample space, with each of its three points having the same probability.

#8 (C)

There is a big difference from the previous problem. Here, we know the gender of a *specific* child (the first born), not of *one of the children*. The known event is now $B = \{GB, GG\}$, with probability $\frac{1}{2}$, and the event that both are girls is a subset of B, $A = \{GG\}$, and has probability $\frac{1}{4}$. Hence, the conditional probability $P[A|B] = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$.

15

This is an example of inductive reasoning, employing Bayes' Rule. Let S be the event "a randomly chosen woman smokes", and E the event that a randomly chosen woman has an ectopic pregnancy.

We are told that ($S^c = \Omega \setminus S$ is the event that a randomly chosen woman is a non smoker)

- $P[S] = .32$
- $P[E|S] = 2 \cdot P[E|S^c]$

and we are requested to evaluate $P[S|E]$.

Applying Bayes' rule, we have

$$P[S|E] = \frac{P[E|S]\,P[S]}{P[E|S]\,P[S] + P[E|S^c]\,P[S^c]} \tag{1}$$

We know that $P[S^c] = 1 - P[S] = .68$, and that $P[E|S] = 2P[E|S^c]$. Substituting the latter in (1), and dividing both numerator and denominator by $P[E|S^c]$, we have

$$P[S|E] = \frac{2 \cdot .32}{2 \cdot .32 + .68} = .48485$$

#16

Let C be the event of a Cesarean section. We have that $P[C] = .15$. Let S be the event that the child survives birth. We are told that $P[S] = .98$, and that $P[S|C] = .96$. We are asked for $P[S|C^c]$.

Clearly,

$$P[S] = P[S \cap C] + P[S \cap C^c] = P[S|C]\,P[C] + P[S|C^c]\,P[C^c]$$

and, plugging in our numbers, this means

$$.98 = .96 \cdot .15 + P[S|C^c] \cdot .85$$

or

$$P[S|C^c] = \frac{.98 - .96 \cdot .15}{.85} = .98353$$

21

Consider the events H (husband earns more than \$25,000), and W (wife earns more than \$25,000). The table means

$$P[H \cap W] = \frac{54}{500}$$

$$P[H \cap W^c] = \frac{198}{500}$$

$$P[H^c \cap W] = \frac{36}{500}$$

$$P[H^c \cap W^c] = \frac{212}{500}$$

$$P[H] = \frac{198+54}{500} = \frac{252}{500}$$

$$P[W] = \frac{36+54}{500} = \frac{90}{500}$$

It is now easy to answer:

a $P[H^c] = 1 - \frac{252}{500} = .496$

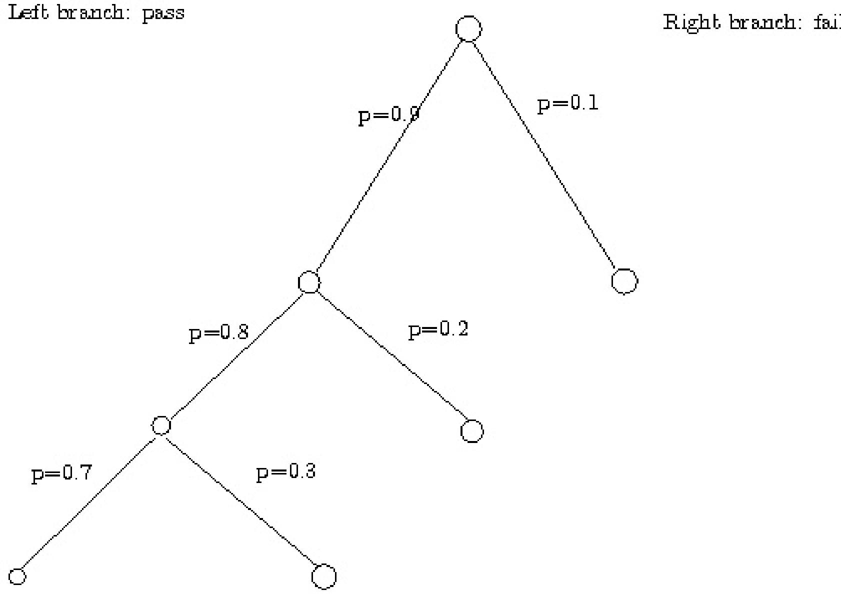
b $P[W\,|H] = \frac{P[H \cap W]}{P[H]} = \frac{54}{252} = .21429$

c $P[W\,|H^c] = \frac{P[W \cap H^c]}{P[H^c]} = \frac{36}{248} = .145$

Note that the last two answers do not have any specific relationship with each other: they refer to two different conditions, and hence result in unrelated probabilities. They definitely do not add up to one!

12 (C)

It is useful to represent the situation as a tree: each node represents an exam, with two branches departing, depending on whether the graduate passes or fails. We can attach to each branch the probability of taking it, *conditioned on the path above it:*



(We'll see many other examples where such a picture can be helpful). Now, if we call the events of passing exams 1, 2, and 3 E_1, E_2, E_3, we have

$$P[E_1 \cap E_2 \cap E_3] = P[E_3 | E_1 \cap E_2] P[E_1 \cap E_2] =$$

$$P[E_3 | E_1 \cap E_2] P[E_2 | E_1] P[E_1]$$

In other words, the probability of reaching the lowest rightmost "leaf", is given by the product of the conditional probabilities of the branches leading to that leaf.

a We just found that the probability of passing all three exams is $.9 \cdot .8 \cdot .7 = .504$

b From the picture, we see that

$$P[E_1^c] = .1$$

$$P[E_2^c] = P[E_1] P[E_2^c | E_1] = .9 \cdot .2 = .18$$

$$P[E_3^c] = P[E_1] P[E_2 | E_1] P[E_3^c | E_1 \cap E_2] = .9 \cdot .8 \cdot .3 = .216$$

Since she cannot fail more than one exam, these three events are disjoint, and their union represents the event we are conditioning to. The event of failing the second exam is a subset of this, so the intersection with it is, again, E_2^c. All in all, the conditional probability we are looking for is

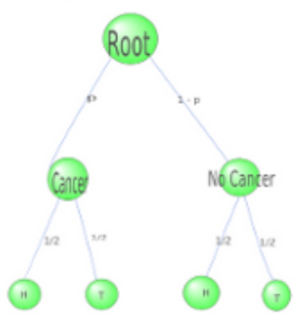
$$\frac{.18}{.1+.18+.216} = .3629$$

27(C)

In a way, the "correct" answer depends on what you are asking precisely. As stated, the question seems to be "given a randomly chosen car, how many workers rode on it?", and hence the correct procedure is *2*. Procedure *1.* answers the question "given a randomly chosen worker, with how many co-workers did he/she share a ride?" - which is a different question, and has, in general, a different answer (big carpools enhance the probability of picking a worker who has many co-riders, but that does not necessarily mean that many cars are pooled).

#31 (C)

We can proceed in a number of ways. For example, we can use a tree diagram, like the following one:



(for typographical reason the probability called α in the problem is denoed by$p$ in this diagram).

Alternatively, one can contruct a picture of a sample space, with four events, C, C^c, H, T that intersect to form four atoms, which we can call CH, CT, C^cH, C^cT. We have $P[C] = \alpha = 1 - P[C^c]$, and $P[H] = P[T] = \frac{1}{2}$. Finally, C is independent of H and T, as is C^c. In fact, it is more elegant to consider two independent random variables, each taking on two values – essentially the indicator functions, respectively, of C and H.

Given independence,

$$P[CH] = \frac{\alpha}{2}, P[CT] = \frac{\alpha}{2}, P[C^cH] = \frac{1-\alpha}{2}, P[C^cT] = \frac{1-\alpha}{2}$$

Given the rule, the event "the doctor does not call", call it N, is the complement of the only case in which he would call, which is C^cH. Hence, $P[N] = 1 - P[C^cH] = 1 - \frac{1-\alpha}{2} = \frac{2-1+\alpha}{2} = \frac{1+\alpha}{2}$ (it is also, of course, the sum of the probabilities of the other three atoms).

We are looking for $\beta = P[C|N] = \frac{P[C\cap N]}{P[N]}$. Now, since $N = CH \cup CT \cup C^cT$, it is clear that $C \subset N$, so that $P[C \cap N] = P[C] = \alpha$. Summing it all up,

$$\beta = P[C|N] = \frac{P[C\cap N]}{P[N]} = \frac{P[C]}{P[N]} = \frac{\alpha}{\frac{1+\alpha}{2}} = \frac{2\alpha}{1+\alpha}$$

To compare α with β, we calculate

$$\alpha - \beta = \alpha - \frac{2\alpha}{1+\alpha} = \frac{\alpha+\alpha^2-2\alpha}{1+\alpha} = \frac{\alpha^2-\alpha}{1+\alpha} = \frac{\alpha(\alpha-1)}{1+\alpha} < 0$$

since $0 < \alpha < 1$. While a call can only bring good news, not receiving a call, quite intuitively, means that the probability of bad news has increased.

#48

We are in the region of the questions of type "three prisoners", "Monte Hall", etc. You would think that the probability should be $\frac{1}{2}$, but look closer...

We are going to solve the problem in the "automatic way", i.e., by applying Bayes' Rule. A classroom note will be posted illustrating more "hands-on" solutions that may (or may not) help figure out the mechanism of these problems.

We have that two events are SS and SG, describing the two cabinets. We found S, and ask for the conditional probability of SS, given S. Now, $P[S|SS] = 1$, $P[S|SG] = \frac{1}{2}$. Also, we assume that the choice of cabinet was "random", i.e. both have probability $\frac{1}{2}$ (this assumption is crucial in determining the correct answer - see the classroom note).

$$P[SS|S] = \frac{P[S|SS]\,P[SS]}{P[S]}$$

and $P[S] = P[S|SS]\,P[SS] + P[S|G]\,P[G] = 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$. Substituting

$$P[SS|S] = \frac{1 \cdot \frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

#49

This is an example of application of Bayes' Rule, as well as an illustration of the power of "prior beliefs". Consider the event C, that a random patient has prostate cancer. We have, in the first version, that, according to the physician, $P[C] = 0.7$. Let T be the event that the test was positive (i.e., showed an elevated PSA level). We are told that $P[T|C] = 0.268$, and $P[T|C^c] = 0.135$.

We apply Bayes' Rule to find the "reversed" conditional probabilities, which are the ones we are asked for:

$$P[C|T] = \frac{P[T|C]\,P[C]}{P[T]} = \frac{P[T|C]\,P[C]}{P[T|C]\,P[C] + P[T|C^c]\,P[C^c]} =$$

$$= \frac{0.268 \cdot 0.7}{0.268 \cdot 0.7 + 0.135 \cdot 0.3} \approx 0.82$$

$$P[C|T^c] = \frac{P[T^c|C]\,P[C]}{P[T^c]} = \frac{P[T^c|C]\,P[C]}{P[T^c|C]\,P[C] + P[T^c|C^c]\,P[C^c]} =$$

$$= \frac{0.732 \cdot 0.7}{0.732 \cdot 0.7 + 0.865 \cdot 0.3} \approx 0.66$$

Reversing the "prior probabilities" of the patient having cancer, we find

$$P[C|T] = \frac{0.268 \cdot 0.3}{0.268 \cdot 0.3 + 0.135 \cdot 0.7} \approx 0.46 \qquad P[C|T^c] = \frac{0.732 \cdot 0.3}{0.732 \cdot 0.3 + 0.865 \cdot 0.7} \approx 0.27$$

52 (C)

Let A be the event the student is accepted: $P[A] = .6$. We are also given a table for all the conditional probabilities for acceptance and rejection, for each day of the week. Actually, we have to be careful in reading the table. What it is saying is that we have disjoint events M, T, W, Th, F, nm, indicating the day of the week in which the mail might arrive (nm meaning no mail arrives in the week), forming a partition of the sample space, and two disjoint events A, R (accepted/rejected) that also form a partition of the sample space. We have

$$P[M|A] + P[T|A] + P[W|A] + P[Th|A] + P[F|A] + P[nm|A] = 1$$

and similarly when we condition on R. For example, this means that

$$M^c = T \cup W \cup Th \cup F \cup nm$$

etc. Also, we find that $P[nm|A] = 1 - .85 = .15, P[nm|R] = 1 - .6 = .4$, and $P[nm] = P[nm|A]\,P[A] + P[nm|R]\,P[R] = .15 \cdot .6 + .4 \cdot .4 = .25$

Applying "total probabilities", Bayes' Rule, etc. we have

a $P[M] = P[M|A]\,P[A] + P[M|R]\,P[R] = .15 \cdot .6 + .05 \cdot .4 = .11$

b $P[T|M^c] = \frac{P[T \cap M^c]}{P[M^c]}$, but $T \subseteq M^c$, so the fraction is

$$\frac{P[T]}{P[M^c]} = \frac{P[T|A]\,P[A] + P[T|R]\,P[R]}{P[M^c]} = \frac{.2 \cdot .6 + .1 \cdot .4}{1 - .11} = .17978$$

c $P[A|M^c \cap T^c \cap W^c] = \frac{P[M^c \cap T^c \cap W^c|A]\,P[A]}{P[M^c \cap T^c \cap W^c]}$ Now, $M^c \cap T^c \cap W^c = Th \cup F \cup nm$, so that

$$[M^c \cap T^c \cap W^c|A]\,P[A] = P[Th|A]\,P[A] + P[F|A]\,P[A] + P[nm|A]\,P[A]$$

$$P[M^c \cap T^c \cap W^c] = P[Th|A]\,P[A] + P[F|A]\,P[A] + P[nm|A]\,P[A] +$$

$$+ P[Th|R]\,P[R] + P[F|R]\,P[R] + P[nm|R]\,P[R]$$

$$P[A|M^c \cap T^c \cap W^c] = \frac{.6 \cdot (.15 + .1 + .15)}{6 \cdot (.15 + .1 + .15) + .4 \cdot (.15 + .2 + .4)} = .4444$$

d This a straight application of Bayes' Rule:

$$P[A|Th] = \frac{P[Th|A]\,P[A]}{P[Th]} = \frac{.15 \cdot .6}{.15 \cdot .6 + .5 \cdot .4} = .6$$

e Again, apply Bayes' Rule, and find

$$P[A|nm] = \frac{P[nm|A]}{P[nm]} P[A]$$

From the table, we get that

$$= \frac{.15 \cdot .6}{.15 \cdot .6 + .4 \cdot .4} = .36$$

53 (C)

We consider the following events: E_i "component #i is working" ($i = 1, 2, \ldots, n$); E "the system is working".

In this approach to reliability theory, the system is decomposed in a number of independent components, and the overall system state is defined by configurations of the components. For example, a "parallel system" is a system that will be "on" when at least one component is "on", and "off" if all components are "off". Note that "components" need not be physical parts - they might be just a computational device masking a very different physical architecture.

Now, the probability of being "on" for the system, i.e., $P[E]$, is to be calculated from the probability of being "on" for each component - and here is where the assumption of independence comes in handy. For our parallel system, it is simplest to evaluate $P[E] = 1 - P[E^c]$, and

$$P[E^c] = P\left[\bigcap_{i=1}^{n} E_i^c\right] = \prod_{i=1}^{n} P[E_i^c] = \prod_{i=1}^{n}(1 - p_i)$$

if $p_i = P[E_i]$. For a large n, this formula is enormously simpler than the full product rule, involving very many, and very long conditional probabilities.

Note that, in our specific case, $P[E] = 1 - \frac{1}{2^n}$, which is very close to 1, as soon as n is large - a parallel system is very reliable, because it is very redundant. Of course, once n is big enough, the gain in reliability of going from n to $n+1$ is negligible, compared to the cost of adding yet one more redundant component.

As for the specific question in this problem, we may apply Bayes' Rule:

$$P[E_1|E] = \frac{P[E|E_1]\,P[E_1]}{P[E]}$$

and this is easy: $P[E|E_1] = 1$, since 1 is enough to keep the whole system going. Hence

$$P[E_1|E] = \frac{1 \cdot \frac{1}{2}}{1 - \frac{1}{2^n}} = \frac{1}{2 - \frac{1}{2^{n-1}}} = \frac{2^{n-1}}{2^n - 1}$$

which is very close to $\frac{1}{2}$, as soon as n is reasonably large. For reference purposes, note that $\frac{1}{2^5} = 0.031$, $\frac{1}{2^{10}} = 9.8 \cdot 10^{-4}$, etc.

#55

The model that is suggested here is that there are four events, F_1 "student selected is a freshman boy", F_2 "student selected is a freshman girl", S_1 "student selected is a sophomore boy", and S_2 "student selected is a sophomore girl". We also know that, if N is the total number of students in the class,

$$P[F_1] = \frac{4}{N}, P[F_2] = \frac{6}{N}, P[S_1] = \frac{6}{N}, P[S_2] = \frac{N-16}{N}$$

The basic events are disjoint, so

$$P[F_1 \cup F_2] = \frac{10}{N}, P[S_1 \cup S_2] = \frac{N-10}{N}, P[F_1 \cup S_1] = \frac{10}{N}, P[F_{12} \cup S_2] = \frac{N-10}{N}$$

We now want to make sure that $B = F_1 \cup S_1$, and $B^c = G = F_2 \cup S_2$ are independent of $F = F_1 \cup F_2$, and $F^c = S = S_1 \cup S_2$. It is only necessary to check one independence relation, since we know that if A is independent of B, it is also independent of B^c. We need, for instance,

$$P[(F_1 \cup F_2) \cap (F_1 \cup S_1)] = P[F_1 \cup F_2] P[F_1 \cup S_1]$$

Now,

$$P[(F_1 \cup F_2) \cap (F_1 \cup S_1)] = P[F_1] = \frac{4}{N}$$

while

$$P[F_1 \cup F_2] P[F_1 \cup S_1] = \frac{10}{N} \cdot \frac{10}{N} = \frac{100}{N^2}$$

and

$$\frac{4}{N} = \frac{100}{N^2}$$

if

$$1 = \frac{25}{N}$$

or $N = 25$. Since we already had accounted for 16 students, we need 6 sophomore girls.

#57 (C)

This model is the starting point for the so-called "binomial model" for stock prices (the name comes from the "binomial distribution", which, as we will soon see, characterizes the probabilities for the ending price of a stock after n days. As naive as this model clearly is, it is widely used for computational purposes in the financial industry, when complex financial instruments, like complicated options, need to be priced.

Of course, this is also the same model that we meet in the "gambler's ruin" problem, and is also the "random walk" model, where a particle (or some other moving item) moves "up" and "down" or "left" and "right", etc. one step at the time, with probabilities p and $1-p$, and the direction of each step is independent of whatever happened until that point.

Let's call U_i the event that our stock moved up at the i-th step, and D_i the event it moved down. We have $P[U_i] = p, P[D_i] = 1-p$. Consider now the specific questions.

a For the stock to be back to the starting price after two days, it had to go either first up, then down, or first down, then up. The first event is $U_1 \cap D_2$, and the second $D_1 \cap U_2$. They are disjoint, and both have probability $p(1-p)$, ***because we assumed that price changes on different days are independent.*** Hence, the probability we were looking for is $2p(1-p)$. Note that the function $p(1-p) = p - p^2$, between 0 and 1, goes from 0 at $p = 0$, to its maximum $\frac{1}{4}$, at $p = \frac{1}{2}$, then back to 0 for $p = 1$. Hence, the highest probability of returning to the starting point occurs when up- and down-movements are equally likely (a "symmetric random walk"), and is equal to $\frac{1}{2}$.

b Using the same logic, we need two ups and one down for this event to occur, and the one down could occur at the first, second, or third day. All three combinations have the same probability: $p^2(1-p)$, and they are disjoint, hence we have a probability of $3p^2(1-p)$. Note how this function too goes from 0 to a maximum, then back to 0, as p ranges from 0 to 1, but now the maximum is at $2p - 3p^2 = 0$, i.e. ($p = 0$ corresponds to a minimum, when we look at this function over all reals) $p = \frac{2}{3}$, when the probability is $\frac{4}{9}$.

c We know already that the event in question (stock up 1 over three days), call it E, is made up of the union of three disjoint events: $D_1 \cap U_2 \cap U_3$, $U_1 \cap D_2 \cap U_3$, and $U_1 \cap U_2 \cap D_3$. We are asked to evaluate

$$P[\{U_1 \cap D_2 \cap U_3\} \cup \{U_1 \cap U_2 \cap D_3\} \,|\, \{U_1 \cap D_2 \cap U_3\} \cup \{U_1 \cap U_2 \cap D_3\} \cup \{D_1 \cap U_2 \cap U_3\}]$$

This is equal to

$$\frac{P[\{U_1 \cap D_2 \cap U_3\} \cup \{U_1 \cap U_2 \cap D_3\}]}{P[E]} = \frac{2p^2(1-p)}{3p^2(1-p)} = \frac{2}{3}$$

(which was pretty easy to guess anyway, since E is made up of three equiprobable events, and we are looking at two of them).

58

We assume that successive flips are independent, and identical, but that the probability of getting H is unknown, say p.

The procedure indicated in the experiment consists in throwing two times, and ignore the result if it is HH or TT. If it is HT or TH we keep the second result.

a Since each toss is independent of all the previous ones, it doesn't matter how many times we had to go back to step 1. Once our two tosses come out HT or TH, they happen with probability $p(1-p)$, and $(1-p)p$ respectively, no matter what happened before. Hence we pick H or T with the same probability. To be specific, we have a conditional probability of

$$\frac{p(1-p)}{p(1-p)+(1-p)p} = \frac{1}{2}$$

b The second procedure works like this: we toss the coin n times, with the first $n-1$ being all equal, and the n-th different. This means that we always pick the result that is opposite from whatever the result of the first toss was. In other words, we pick H wit probability $1-p$, and T with probability p - no improvement on the direct method of picking the first outcome, with the probabilities simply switched.

61 (C)

This is the standard model for genetic transmission of characters, through gene inheritance. In this model, $P[AA] = P[Aa] = P[aA] = P[aa] = \frac{1}{4}$, and you receive one gene from each parent's pair, with probability $\frac{1}{2}$ each. Note, however, that there is no purely mathematical reason to consider the two configurations aA and Aa as distinct - this is a biological question (how is this information stored?). Very much like the decision to consider HT and TH distinct outcomes for the coin toss - appropriate for real coins, but (as it turns out) inappropriate for bosons, like protons - is not a mathematical decision per se, since it simply corresponds to the choice of one model over another.

Consider now the standard model. In this problem, we have two children, one of which is of type aa. Since the parents are only carriers, they must be Aa or aA. The other child is not aa, so he/she can be aA, Aa, AA. Note that, given the genetic structure of the parents, a priori, each combination was equiprobable. The second child has thus $\frac{2}{3}$ probability of being a carrier.

Note that the "non-standard" model (aA is not different from Aa) is not very significant here: once your parents are both healthy carriers, if you are healthy, you still have the same $\frac{2}{3}$ probability of being a carrier, since the options are three out of the 4 possible ones: a from father + A from mother, a from mother + A from father, A from both (since the fourth, a from both, has been excluded).

a This second child marries with a person known to be Aa or aA. The spouse will pass an a gene with probability $\frac{1}{2}$. The child will too - conditional on being a carrier, which has probability $\frac{2}{3}$: the probability of transmitting an a gene is thus $\frac{2}{6} = \frac{1}{3}$. The two transmissions being independent, the resulting offspring will be an albino with probability $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ (instead of $\frac{1}{4}$, when both parents are *known* to be carriers). In standard "total probability" formulation, if D is the event "the newborn child is albino", and C the event "the child of the original couple is a carrier"

$$P[D] = P[D|C]P[C] + P[D|C^c]P[C] = \frac{1}{4}\cdot\frac{2}{3} + 0\cdot\frac{1}{3} = \frac{1}{6}$$

b Now, we are considering D_2, given D_1^c (the subscripts referring to the birth order):

$$P[D_2|D_1^c] = \frac{P[D_1^c \cap D_2]}{P[D_1^c]}$$

We know that $P[D_1^c] = \frac{5}{6}$, and

$$P[D_1^c \cap D_2] = P[D_1^c \cap D_2|C]P[C] + P[D_1^c \cap D_2|C^c]P[C^c] =$$

$$= \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{2}{3} + 0\cdot\frac{1}{3}$$

since each child receives genes independently, and, provided both parents are carriers, there is $\frac{1}{4}$ possibility of the second child getting a from both, and thus (independently) $\frac{3}{4}$ of not getting it for the first child. Summing up,

$$P[D_2|D_1^c] = \frac{\frac{1}{8}}{\frac{5}{6}} = \frac{3}{20}$$

66 (C)

This relay picture is often used as a metaphor for reliability model, where each relay corresponds to a component, "closed" corresponds to "working", and "current flows" maps to "system is working".

a This system works if, first of all, relay 5 is closed , and then, if at least one of the two sequences 1-2 and 3-4 (each a "series" system) is working (the two sequences are "in parallel"). The two sequences are working, respectively, with probability p_1p_2, and p_3p_4. Their parallel arrangement, will work with probability $1-(1-p_1p_2)(1-p_3p_4)$. Overall,

$$p_5(1-(1-p_1p_2)(1-p_3p_4)) = p_5(p_1p_2 + p_3p_4 - p_1p_2p_3p_4)$$

b The hint suggests to compute the probability of the system working, say $P[W]$ from

$$P[W|3]P[3] + P[W|3^c]P[3^c] = P[W|3]p_3 + P[W|3^c](1-p_3)$$

If 3 is closed, the system will *not* work only if both 1 and 2 are open, or, if at least one of them is closed, both 4 and 5 are open. I.e., it will work as a series of two relays, the first working with probability $1-(1-p_1)(1-p_2)$, the second with probability $1-(1-p_4)(1-p_5)$. Overall,

$$P[W|3] = (1-(1-p_1)(1-p_2))(1-(1-p_4)(1-p_5))$$

If 3 is open, then we have a straight parallel of two series lines: the upper path is available with probability $p_1 p_4$, and the second with probability $p_2 p_5$. The parallel between the two works with probability

$$1 - (1 - p_1 p_4)(1 - p_2 p_5)$$

Combining all the above, we get

$$P[W] = p_3 (1 - (1 - p_1)(1 - p_2))(1 - (1 - p_4)(1 - p_5)) + (1 - p_3)(1 - (1 - p_1 p_4)(1 - p_2 p_5))$$

Remark The solution manual ignores the hint for question (b) altogether, and simply lists all combinations that result in a working system, calculating the probability of each. The result is no different, but is more cumbersome to evaluate. Note also that problem (a) represents an example of a "parallel-series" circuit - i.e., a combination of parallels, and series blocks. Not all circuits can be described as such, but the hint for (b) (which is *not* a parallel-series circuit) indicates how, through conditioning on a strategically chosen relay, we can often reduce the problem to a combination of parallel-series circuits, which is how we solved the problem here.

67 (C)

k out of n is another popular model for the reliability of systems.

a To work 2 components (at least) must work. Not to work, 3 or all must fail. The latter happens with probability ($q_i = 1 - p_i$)

$$q_1 q_2 q_3 p_4 + q_1 q_2 q_4 p_3 + q_2 q_3 q_4 p_1 + q_1 q_2 q_3 q_4 = p_f$$

Hence the required probability is $1 - p_f$

b This is much the same: we compute p_f, which now corresponds to 4 or all 5 components failing, and then take $1 - p_f$

Remark The solution manual goes for the direct calculation (probability of 2 or 3 or 4 working, for (a), and probability of 3, 4 or 5 working for (b)). The result is obviously the same, but the expression is shorter when you check 2 conditions instead of 3

c Things get quickly complicated when the p_i's are different, as the number of components increases. When all components have the same reliability, p, then we are, effectively, tossing n coins, and looking for the probability of $k, k+1, \ldots, n$ to come up "heads" (or whatever we call the event that has probability p). This is, in formulas,

$$\sum_{j=k}^{n} \binom{n}{j} p^j (1 - p)^{n-j}$$

(we counting all strings of j symbols 1, and $n-j$ symbols 0, and that's the number of ways to arrange them - refer to chapter 1 for the details, but if you are familiar with Newton's binomial formula

$$(a+b)^n = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) a^j b^{n-j}$$

the coefficient in front of the product does just that: it counts the way you can arrange j symbols of one type - here a - and $n-j$ of another type - here b.)

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Note that we are not assuming that $E = F^c$. As we repeat the experiment, we wait for either one to occur. Once one of the two has occurred, which one was it? Well, we are looking at, say,

$$P[E \,|E \cup F] = \frac{P[E \cap (E \cup F)]}{P[E \cup F]} = \frac{P[E]}{P[E] + P[F]}$$

since the two events are incompatible.

81

This is a direct application of the gambler's ruin problem (Example 4j, p. 90). The investor is playing a game, with probability p of winning, and probability $1-p$ of losing ($p = .55$). Formula (4.5) on p. 92 tells us that the probability of ending up "with all the money" (reaching level N), when starting from level i is

$$\frac{1 - \left(\frac{1-p}{p}\right)^i}{1 - \left(\frac{1-p}{p}\right)^N} \tag{2}$$

(we are in the case $p \neq \frac{1}{2}$). The "zero" level is given when the stock (which is today at 25) hits 10. The "N" level is when it reaches 40. So $N = 40 - 10 = 30$, and we are starting at $i = 25 - 10 = 15$. The formula (2) gives us

$$\frac{1 - \left(\frac{.45}{.55}\right)^{15}}{1 - \left(\frac{.45}{.55}\right)^{30}} = \frac{1 - \left(\frac{.45}{.55}\right)^{15}}{\left(1 - \left(\frac{.45}{.55}\right)^{15}\right)\left(1 + \left(\frac{.45}{.55}\right)^{15}\right)} = \frac{1}{1 + \left(\frac{.45}{.55}\right)^{15}} = \frac{1}{1 + \left(\frac{9}{11}\right)^{15}} = .953$$

This is not surprising: our investor is acting like a casino, with a good bias in its favor (55% is better than most plays at roulette) against a player that is starting at the middle of its allowed range (he will quit when the game will have deviated from the start by 15, up or down). As we know form the gambler's ruin problem, even with less outrageous odds, the casino has always a very large probability of coming out the winner in such a game.