Math 394B & C

Solutions to Homework Problems

Let us be up front about some of the following problems (for example, the poker odds problems). Your instructor is not terribly good at solving them either, but, not being a gambler, doesn't feel bad about it at all. These tricky combinatorial problems are here to show how they are done, and what kind of results you can get, but they are very marginal - unless you are planning a gambling career, or, for that matter, a combinatorial career. Combinatorics is a very complex field, with several fascinating applications, but it has minimal overlap with probability. The next assignments will be richer in core issues to the class.

#1

Let us call the marbles, according to their color, **R**,**G**, and **B**. The "tightest" sample space that we can consider will have to account for all possible outcomes when we only look at the color of the balls we pick. The possible outcomes differ, depending on what we do with the first ball we pick. We have thus the two following cases:

1. The ball we pick first is put back in the box, before we pick the second. Hence, the first part of the experiment and the second look the same, and both allow for any of the balls to be chosen: we can use the following space

$$\Omega = \{RR, RG, RB, GG, GR, GB, BB, BR, BG\}$$

   Note that $\#\Omega = 9 = 3^2$

2. In the second scheme, the first ball is not put back, hence, the outcomes where both extractions yield the same color are excluded. The sample space will now be

$$\Omega = \{RG, RB, GR, GB, BR, BG\}$$

   and $\#\Omega = 6 = 3 \cdot 2$

The numbers we found for $\#\Omega$ are well justified. In the first case, we have 3 possible outcomes for the first extraction, and, corresponding to each of these three, three more for the second - hence $3 \cdot 3 = 3^2$. In the second, once a ball is extracted, the possible outcomes for the second extraction are only 2 - hence $3 \cdot 2$.

More generally, if we had n different balls, and extracted k of them with the first rule (putting the extracted ball back every time), the number of possible results would be n^k.

In the second case, when we do not put the ball back, we have, necessarily $k \leq n$ (once we have extracted n balls, nothing is left), and the number of possible outcomes is counted by noting that we have n possibilities at the first extraction, $n-1$ at the second, $n-2$ at the third, and so on, until the $k-$th, when we have $n-(k-1)$ (we have already taken $k-1$ balls away. Hence the number of possible results is $n(n-1)(n-2)\ldots(n-k+2)(n-k+1)$.

Details on this type of calculations can be found in chapter 1 of the book.

#2

The problem at hand is logically tricky, in that no specific upper number of trials has been given. In principle, the experiment should go on until a 6 appears, and nobody is ruling out that it won't appear (in this particular experiment) in a thousand years... In practice, experiments like this do not continue "indefinitely". First of all, it would be highly unusual for a 6 never to appear in several dozens or hundreds of tosses. Second, if we had not got a 6 after a few hours, we would certainly call it quits. But, unless we have a specific threshold in mind (say: "I will stop after 1000 tosses, no matter what"), there is no natural way to determine an upper limit. Since we do not have a natural bound, the mathematician in us will choose to allow "infinite sequences" of tosses - where what is meant is "unlimited", rather "infinite", in the sense that we are always willing to go to toss $n+1$ after we have gone through n tosses, even if we will not *actually* make *infinitely many* tosses.

We have a choice of sample space, depending on whether we want to focus on the present experiment only (which only checks whether we have a 6 or not), or wish to allow for further analysis (like what numbers came up before the 6 ended our sequence).

In the first case, we only need to check whether we had a "success" (6 came up), or not. Denote a "success" by 1, and a "failure" (some other number came up) by 0. Then our possible outcomes are sequences of 0s, terminated by a 1, like

$$\{1\}, \{01\}, \{001\}, \{0001\}, \{00001\}, \ldots$$

and so on. In principle, we are dealing here with infinitely many sequences, one for each toss at which we finally get a 6. In the notation of the book, $E_1 = \{1\}, E_2 = \{01\}, \ldots$. Here, our sample space has each event E_k as a singleton event.

We could however choose Ω to keep track of the specific results of each toss before a 6. In this case, elements ω would look like, say, $\{1, 4, 2, 2, 4, 5, 1, 6\}$, or

$\{2,3,3,1,5,6\}$, etc. The event E_1 would include only the element $\{6\}$, but E_2 would consist of $\{1,6\},\{2,6\},\{3,6\},\{4,6\},\{5,6\}$, and so on.

Finally, note that, whatever choice of Ωwe make, we have to consider the possibility that a 6 *never* appears. This would correspond to the only actually infinite sequence in our space: in the first notation, an infinite sequence of zeros. In the second, the (infinitely) many sequences made of $1,2,3,4,5$ only. If we call the corresponding event E_∞, we can observe that $\bigcup_{k=1}^{k=\infty} E_k$ means "for some k, $\omega \in E_k$". Hence

$$\Omega = \left(\bigcup_{k=1}^{k=\infty} E_k \right) \bigcup E_\infty$$

or

$$E_\infty = \Omega \setminus \left(\bigcup_{k=1}^{k=\infty} E_k \right) = \left(\bigcup_{k=1}^{k=\infty} E_k \right)^c$$

#5 (C)

A possible sample space (let's call it the minimal possible), will record all possible "states" for the system, as defined in the problem: all 5-vectors of 0s and 1s, where a 1 in the $j-$th position means that component j is working, and a 0 means it is failing.

a We could enumerate all possibilities:

$$\{0,0,0,0,0\},\{1,0,0,0,0\},\{0,1,0,0,0\},\{0,0,1,0,0\},\ldots$$

$$\ldots,\{0,1,1,1,1\},\{1,1,1,1,1\}$$

but that would be *very* awkward (and think of the time involved, if we were looking at, say, 100 components). We can, however, calculate the total number of 5-vectors needed. In fact, each position can take 2 values, hence two positions an take, overall, $2\cdot 2 = 4$ values, three can take $2\cdot 2\cdot 2 = 2^3 = 8$ values, and so on. For 5 positions, we have a total of $2^5 = 32$ possible combinations of 0s and 1s. If we had, say, 100 components, the number would be 2^{100}, which is really too big for simple "counting" to work (in fact $2^{100} \simeq 1.3 \cdot 10^{30}$, which seems big enough).

b Let us denote by x the value in a position of which we do not specify whether it is a 0 or a 1. The case where 1 and 2 work (and we don't care what the others do) would be all vectors of the form $\{1,1,x,x,x\}$. The case where 3 and 4 work would be all vectors of the form $\{x,x,1,1,x\}$, and the case where 1,3,5 work would be $\{1,x,1,x,1\}$. There is plenty of overlap between these events (call them, in order, A_1, A_2, A_3. It is true that $W = A_1 \bigcup A_2 \bigcup A_3$. But, for instance,

$$A_1 \bigcap A_2 = \{1,1,1,1,x\}$$

$$A_1 \bigcap A_3 = \{1, 1, 1, x, 1\}$$

$$A_2 \bigcap A_3 = \{1, x, 1, 1, 1\}$$

$$A_1 \bigcap A_2 \bigcap A_3 = \{1, 1, 1, 1, 1\}$$

So, how many "points" belong to W? Well, $\#A_1 = 2^3$, because we have 3 "free" variables, $\#A_2 = 2^3$ as well, and $\#A_3 = 2^2$. However, if we simply added these numbers together, we would count the intersections twice, and we need to subtract $A_1 \bigcap A_2 = 2$ $\#A_1 \bigcap A_3 = 2$, $\#A_1 \bigcap A_2 = 2$. In so doing, we subtracted $\#A_1 \bigcap A_2 \bigcap A_3 = 1$ once too many, so we have to add it back. In fact, we applied, by hand, what is a particular case of the so-called *Bonferroni Formula* (see the book). In the end,

$$\#W = 2^3 + 2^3 + 2^2 - 2 - 2 - 2 + 1 = 2^4 + 2^2 - 5 = 15$$

c Similarly, using our notation in point , A would be made up of all vectors of the form $\{x, x, x, 0, 0\}$, and $\#A = 2^3 = 8$.

d AW is usually denoted by $A \bigcap W$. Looking at our events A_1, A_2, A_3, it is clear that $A \bigcap A_2 = A \bigcap A_3 = \emptyset$. Hence

$$A \bigcap W = A \bigcap \left(A_1 \bigcup A_2 \bigcup A_3\right) =$$

$$= \left(A \bigcap A_1\right) \bigcup \left(A \bigcap A_2\right) \bigcup \left(A \bigcap A_3\right) =$$

$$A \bigcap A_1$$

i.e., all 5-vectors of the form $\{1, 1, x, 0, 0\}$. There are exactly two of them.

#8 (C)

Since we are told that $A \bigcap B = \emptyset$, the additivity axiom tells us that

$$P\left[A \bigcup B\right] = P[A] + P[B]$$

Hence,

a The probability of either A or B occurring is ("or" corresponding to "union") $P[A] + P[B] = .3 + .5 = .8$

b Now we are looking at $P[A \bigcap B^c]$. Reflection shows (you can draw a Venn diagram to convince yourself), that mutual exclusivity of A and B implies $A \subseteq B^c$, and so $A \bigcap B^c = A$. In other words, $P[A \bigcap B^c] = P[A] = .3$

c Since A and B are mutually exclusive, they cannot occur simultaneously, i.e. ("and" corresponding to "intersection"), $A \bigcap B = \emptyset$, and $P[A \bigcap B] = P[\emptyset] = 0$

#9

Let Ωbe the totality of the establishment's customers. The subset A, of those carrying American Express, has relative frequency .24, and the subset V, of those carrying Visa has frequency .61. We also have that the frequency of $A\bigcap V$ is .11. What we are interesed in is $A\bigcup V$ (American Express *or* Visa):

$$P\left[A\bigcup V\right] = P[A] + P[V] - P\left[A\bigcup V\right] = .24 + .61 - .11 = .74$$

#11 (C)

Call A the set of male cigarette smokers, B the set of male cigar smokers, and S the set of male smokers. Our data says $P[A] = .28, P[B] = .07, P[A\bigcap B] = .05$

a The set of male nonsmokers is given by $P[A^c\bigcap B^c]$, but, since intersections are not directly computable from the axioms, it might be better to use the fact that

$$A^c\bigcap B^c = \Omega\setminus\left(A\bigcup B\right)$$

so that

$$P\left[A^c\bigcap B^c\right] = 1 - P\left[A\bigcup B\right]$$

and, since

$$P\left[A\bigcup B\right] = P[A] + P[B] - P\left[A\bigcap B\right] = .28 + .07 - .05 = .3$$

$$P\left[A^c\bigcap B^c\right] = 1 - .3 = .7$$

b Here we are looking at $B\setminus A$ (elements in B, but not in A). It is easy to see (use a Venn diagram, if you don't find this obvious) that $B\setminus A = B\setminus(A\bigcap B)$, and $A\bigcap B\subseteq B$, so that

$$P[B\setminus A] = P\left[B\setminus\left(A\bigcap B\right)\right] = P[B] - P\left[A\bigcap B\right] = .07 - .05 = .02$$

#12 (C)

Consider the percentage of students in each language class. We have $P[S] = .28, P[F] = .26, P[G] = .16$ (S for Spanish, F for French, and G for German). We also have $P[S\bigcap F] = .12$, $P[S\bigcap G] = .04$, $P[F\bigcap G] = .06$, and $P[S\bigcap F\bigcap G] = .02$.

a We are looking at the complement of $S\bigcup F\bigcup G$. This probability is most easily calculated by

$$P\left[\left(S\bigcup F\bigcup G\right)^c\right] = 1 - P\left[S\bigcup F\bigcup G\right]$$

and

$$P\left[S\bigcup F\bigcup G\right] = P[S]+P[F]+P[G]-P\left[S\bigcap F\right]-P\left[S\bigcap G\right]-P\left[F\bigcap G\right]+$$

$$+P\left[S\bigcap F\bigcap G\right] = .28+.26+.16-.12-.04-.06+.02 = .5$$

Hence 50% of the student are taking at least one language class, and 50% are taking none.

b We need to subtract from $P[S\bigcup F\bigcup G]$ the probability of the intersections, i.e. we are looking at

$$P[S\cup F\cup G]-P[S\cap F]-P[S\cap G]-P[F\cap G]+2P[S\cap F\cap G]=$$

$$=.5-.12-.04-.06+2\cdot .02=.32$$

c Now, we are looking at events E_1, the first student is in a language class, E_2, the second student is in a language class, and ask for $P[E_1\bigcup E_2]$. Thinking in the spirit of problem (student in place of marbles, and classes instead of colors, with the appropriate variations), we notice that, at least in principle, we should abide by the method used there in point 2: after all, once we pick the first student, the second is picked form the remaining 99 - we don't pick the same student twice. However, this makes the calculation more involved, since, depending on what the first student does, we have different odds for the second! This is a common problem in "sampling", when, for instance in an opinion poll, we pick individuals "at random", but avoid picking the same individual twice, altering the pool for which we extract at every turn. If the original pool is large enough, and the number of individuals picked small enough, it doesn't make much difference whether we "put the individual back in the pool" or not. After all, for example. $\frac{30}{100}=.3$, and $\frac{30}{99}=.30303\ldots$, so the difference is not very significant. If we treat the problem like point 1 in problem , we have a probability of picking a student that takes at least one class of .5 (see point a). The same happens for the second, i.e. 50% of the time we pick first a student that takes a language class, and, for each of these alternatives, 50% of the time we will pick as second a student who takes one class. In other words, we have a 50% chance of picking right away a student taking a language class, and 50% of the remaining 50% of picking the second student taking a language class, when the first doesn't. Overall, $.5+.5\cdot .5=.5+.25=.75$. If we want to take into account the change in composition of the student population after picking the first student, we could argue as follows: the first student will take a class with probability .5. Now, if he does not, the second student is picked from 99, of whom 50 do take a class. Hence, the probability of taking a class is $\frac{50}{99}=.50505\ldots$. The result is thus $.5+.5\cdot .50505=0.7523\ldots$. Thus our first solution is wrong by $.0023\ldots$. Percentage wise, this is an error of $\frac{.0023}{.7523}=.0033557$. Whether an error of 0.3% is "large" or "small" depends, of course, on the specific application we are considering.

#15

Card games provide a wealth of combinatorial problems. To solve them quickly, we need to refer to formulas developed in chapter 1. Specifically, the starting point for these problems is to determine a *finite partition* of the sample space, $\Omega = \bigcup_{i=1}^{n} A_i$ $(A_i \bigcap A_j = \emptyset, i \neq j)$, such that our problem can be represented by assigning the same probability to each $A_i : P[A_i] = \frac{1}{n}$. The minimal sample space in this case would be made up of n points, corresponding to each element in the partition, with $P[\{\omega_i\}] = \frac{1}{n}$. Problems in such a setting are solved by *counting* how many points (or atoms of the partition) fall in the requested event. Moreover, the number of points, n, is determined by *counting* how many atoms make up the partition.

There are 52 cards in a poker deck, and one hand consists of 5 cards picked "at random". It is usually assumed that "at random", in this context, means that the probability of any specific quintuplet is the same for all. How many quintuplets can we build out of 52 cards? Well, we have 52 ways to pick the first card, 51 for the second, and so on. That would indicate $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52!}{47!}$. However, the *order* in which we receive the cards is immaterial: AKQ or KAQ or QKA, etc. all amount to the same triplet. Hence we divide by the number of equivalent quintuplets, i.e. by all rearrangements of 5 cards - 5!. The result is

$$\frac{52!}{47!5!} = \frac{52!}{(52-5)!5!} = \binom{52}{5} = \binom{52}{47}$$

The last symbol is called a *binomial coefficient*, defined for integers $n \geq k$ as

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$$

Assuming all possible $\binom{52}{5}$ hands have the same probability, we have to count how many hands constitute the various cases listed.

a A flush is when all 5 cards are from the same suit: here are 13 cards in each suit, so the number of ways to extract 5 from one suit is $\binom{13}{5}$ (same argument as above). There are 4 suits, so the possible ways to have a flush are $4 \cdot \binom{13}{5}$. Hence, the probability is

$$\frac{4 \cdot \binom{13}{5}}{\binom{52}{5}} = \frac{4 \cdot 1,287}{2,598,960} = .00198$$

b One pair requires two cards to be the same, and the remaining three to be different: given a card number, there are $\binom{4}{2}$ ways to pick two of them, and $\binom{12}{3}$ ways to pick the other three, so that they have all different numbers. There are also 13 different card numbers we can choose. Finally, each of the three numbers that appear in one hand can be picked in 4 different ways. The result is

$$\frac{13\binom{4}{2}\binom{12}{3}\cdot 4\cdot 4\cdot 4}{\binom{52}{5}} \simeq .42$$

c The argument for two pair is similar, with the following choices involved: pick two out of the 13 possible numbers ($\binom{13}{2}$), pick two out of the four available for each pair ($\binom{4}{2}$), and, finally, one (the fifth card) out of the 44 remaining 44 cards ($\binom{44}{1}$), for an end result of

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}} \simeq .048$$

d Three of a kind, is very similar to two of a kind. The same arguments, with the appropriate changes, give

$$\frac{13\binom{4}{3}\binom{12}{2}\cdot 4\cdot 4}{\binom{52}{5}} \simeq .021$$

e Four of a kind is in the same line, and even simpler, since we have to pick four equal cards, and just one out of the remaining 12 numbers (for 48 cards total):

$$\frac{13\cdot 4\cdot 48}{\binom{52}{5}} \simeq .00096$$

(not terribly common, but, hey, over thousands of hands, it is bound to happen).

#16

Dice games are just as popular, and the counting is easier than for card games. Throwing 5 dice, there are 6^5possible outcomes (6 values for each die), and, as above, a "fair" toss would be one where all 6^5 outcomes have the same probability. The next step is to count how many ways there are to get the outcome described:

a For no two dice to show the same number, we have 6 ways for the first die, but only 5 for the second, 4 for the third, etc. (since the previous dice preempt some of the numbers). All in all (you can match the formulas with the numerical results given with your calculator),

$$\frac{1}{6^5} \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \frac{6!}{6^5} \approx 0.093$$

b One pair comes up when we get two equal points out of 5 ($\binom{5}{2}$), which can happen in 6 different ways. Next, the remaining three dice have to come up with different points, which they can in $5 \cdot 4 \cdot 3$ ways:

$$\frac{6 \binom{5}{2} 5 \cdot 4 \cdot 3}{6^5}$$

c To have two pairs from five dice, we can argue like this: We have six choices for the first pair, five for the second, and four for the odd die out. However, each double pair is counted twice this way, so we will divide by two this product. We can have the five dice come up in that way in $\frac{5!}{2!2!}$ different ways (all permutations of five items, with two pairs being equal). Hence, we have a probability of

$$\frac{\frac{6 \cdot 5}{2} \cdot 4 \cdot \frac{5!}{2!2!}}{6^5} = \frac{5 \cdot 5!}{6^4} \approx 0.248$$

d Three alike, much like a pair, with the same argument gives

$$\frac{6 \cdot 5 \cdot 4 \cdot \binom{5}{3}}{6^5}$$

e A full house, implies you pick two different points, and then three out of the 5 dice that are not equal to the pair:

$$\frac{6 \cdot 5 \cdot \binom{5}{3}}{6^5}$$

f Four alike, much like the card poker question, yields

$$\frac{6 \cdot 5 \cdot \binom{5}{4}}{6^5}$$

g Five alike, is straight forward: you have 6 choices for the point coming up:

$$\frac{6}{6^5}$$

#20

Blackjack is a particular favorite of "combinatorialists" in this field, since the deck is shuffled rarely enough so that a player carefully counting cards has at a certain point of the game the odds in his/her favor against the house. That's one reason why you will be quickly taken to the door if management suspects you of counting cards.

The game is played with each player, and the dealer, being dealt two cards. Each payer, in turn, can ask for an extra card. The goal being to reach a specific combination by adding card points (every **K**,**Q**,**J** is worth 10 points) to reach 21, without overstepping it. The dealer is at advantage since he plays last, and is the winner in case of a tie. However, since the deck is not shuffled every time, a careful count of cards will allow a (very) skilled player to evaluate the odds in his or her favor. You are free to check some of the many books on the game to learn the tricks (one is by a fairly well-known mathematician, Ionescu-Tulcea).

Now, a "blackjack" is a combination of ace and 10-point card (10,J,Q,**K**) which is the top combination to win. The probability of being dealt a blackjack is computed by noting that you have 4 ways of being dealt an ace, and 16 of being dealt a 10-point card $(4 \cdot 4)$, while the possible pairs you can be dealt are $52 \cdot 51$. Hence the probability of a blackjack is $2 \cdot \frac{4 \cdot 16}{52 \cdot 51}$ (you can have either the ace or the card being dealt first). The probability of *both* you *and* the dealer being dealt a blackjack is $(P[A] + P[B] - P[A \bigcap B])$

$$2 \cdot \frac{4 \cdot 16}{52 \cdot 51} + 2 \cdot \frac{4 \cdot 16}{52 \cdot 51} - \frac{4 \cdot 4 \cdot 16 \cdot 3 \cdot 15}{52 \cdot 51 \cdot 50 \cdot 49}$$

which is approximately0.094758 Hence, the probability of *neither* you nor the dealer having a blackjack is $1 - 0.094758$ or about 0.90524

#23 (C)

We have that all possible outcomes can be classified as

1. first die is higher
2. second die is higher
3. dice have the same points

Clearly, by symmetry, $P[1] = P[2]$, while $P[3] = \frac{6}{36} = \frac{1}{6}$ (out of the 36 possible outcomes, 6 are of the form (x, x), since $x = 1, 2, \ldots, 6$). Hence, $P[2] = \frac{1}{2}(1 - P[3]) = \frac{1}{2}\frac{5}{6} = \frac{5}{12}$.

It would also be easy, in such a small experiment, to just count the "favorable outcomes": if the second die lands a 2, there are none, if it lands a 3, there is one, and so on: $\frac{1+2+3+4+5}{36}$

#26

Note that the probability of throwing i, for $i = 2, 3, \ldots, 12$ is

$$p_i = \frac{1}{36} \cdot \min\left(\frac{i}{12}, \frac{12-i}{12}\right)$$

Next, the probability of winning, assuming you threw i at the first toss, $P[E_i]$, is computed by observing that

- $P[E_2] = P[E_3] = P[E_{12}] = 0$
- $P[E_7] = P[E_{11}] = 1$
- $P[E_i] = \sum_{n=1}^{\infty} (1-(p_i+p_7))^{n-1} p_i$ for all remaining i's (it's the probability of not throwing either i or 7 in the first succeeding $n-1$ throws, and i at the nth, summed over all possible n)

Thus, the probability of winning is

$$\sum_{i=2}^{12} p_i \cdot P[E_i] = p_7 + p_{11} + \sum_{i \neq 2,3,7,11,12} p_i^2 \sum_{n=1}^{\infty} (1-p_i-p_7)^{n-1} =$$

$$= \frac{1}{6} + \frac{1}{18} + \sum_{i \neq 2,3,7,11,12} p_i^2 \frac{1}{p_i + p_7} =$$

$$\frac{2}{9} + \sum_{i \neq 2,3,7,11,12} p_i \frac{p_i}{p_i + p_7} = \frac{2}{9} + 2 \cdot \frac{1}{36}\left(3\frac{3}{9} + 4\frac{4}{10} + 5\frac{5}{11}\right) \approx 0.493$$

#27 (C)

Again, standard assumptions in this setting are that each of the 10 balls has the same probability of being picked up - i.e. that each ball has individuality, and so that the probability of picking a red ball is $\frac{3}{10}$, while that of picking a black one is $\frac{7}{10}$.

The game is obviously finite, since it stops, at the latest, after all 7 black balls have been drawn. Hence the possible outcomes can be described by listing them as:

R; BR; BBR; $BBBR$; $BBBBR$; $BBBBBR$; $BBBBBBR$; $BBBBBBBR$

A wins in case 1,3,5,7. Now, all these events are far from equiprobable. To find their probability we can argue as follows:

$$P[R] = \frac{3}{10}$$

(we already noted that)

$$P[BR] = \frac{7}{10} \cdot \frac{3}{9} = \frac{21}{90} = \frac{7}{30}$$

because we first pick a black one, and then we have 9 balls left, 3 of which are red.

$$P[BBR] = \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8} = \frac{126}{720} = \frac{7}{40}$$

and so on.

However, we can also think in terms of a larger sample space: consider the space of all possible sequences of 10 extractions of all balls. One point in such a space would be, e.g.

$$BBRBRRBBBB$$

The possible sequences (remember that each ball is individually distinguished, is the number of possible arrangements of 10 items: the first position can be any of the ten, the second any of the nine left after picking the first, the third any of the eight left, etc: number of arrangements ("permutations") equals 10!, defined by

$$n! = n(n-1)(n-2)\dots 3\cdot 2\cdot 1$$

(the formal definition is by induction: $1! = 1; n! = n(n-1)!$, and it turns out to be convenient to agree and define $0! = 1$).

The sequences corresponding to a "win" by A are those where the first R appears in an odd position, after which, we don't care what happens. So, to have an R come as first can be done in 3 ways (one per red ball) $\times$ the number of possible 9-sequences, which we don't care to look at: $3\cdot 9!$. Assuming that all 10! possible sequences are equiprobable, the probability of getting a red ball first would then be $\frac{3\cdot 9!}{10!} = \frac{3}{10}$ $(10! = 10\cdot 9!)$. The first R will be the third ball if the first two are black, which would happen in $7\cdot 6$ possible ways, the third is an R (which has 3 possible ways of happening, and the remaining 7 picks are whatever they may, which would happen in 7! possible ways. Altogether, $7\cdot 6\cdot 3\cdot 7!$. Proceeding like this we end up with the formula

$$\frac{3\cdot 9! + 7\cdot 6\cdot 3, \cdot 7! + 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 5! + 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 3!}{10!}$$

#28 (C)

Note that that, in the first method, the three balls are selected simultaneously, which is the same as selecting them one by one, without putting them back in the urn after selection.

a There are 5 red, 6 blue, and 8 green balls, for a total of 19. Hence, the ways to pick three reds are (the order in which they are picked being irrelevant) $\binom{5}{3}$, three blue are $\binom{6}{3}$, and three green $\binom{8}{3}$, out of a total of $\binom{19}{3}$. Result is

$$\frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}}$$

b Over the $\begin{pmatrix} 19 \\ 3 \end{pmatrix}$ possible outcomes, we have 5 ways to pick a red ball, 6 ways to pick a blue, and 8 ways to pick a green:

$$\frac{5 \cdot 6 \cdot 8}{\begin{pmatrix} 19 \\ 3 \end{pmatrix}}$$

If we replace each ball after selection, probabilities of picking a specific color do not change from the first to the third selection: $\frac{5}{19}$ for red, $\frac{6}{19}$ for blue, and $\frac{8}{19}$ for green. Hence

a We have

$$\left(\frac{5}{19}\right)^3 + \left(\frac{6}{19}\right)^3 + \left(\frac{8}{19}\right)^3$$

b We have all events of the form $R \bigcap B \bigcap G$ in all orders. However, since each selection is just like the first, we have that the probability of each possible sequence of three different colors is

$$\boxed{\frac{\mathbf{5 \cdot 6 \cdot 8}}{\mathbf{19^3}}}$$

There are 6 different arrangements (**RBC**,**RCB**,**BRC**,**BCR**,**CRB**,**CBR**), and hence the final result is

$$\boxed{\mathbf{6 \cdot \frac{5 \cdot 6 \cdot 8}{19^3} \approx 0.21}}$$

#47 (C)

This a variation of a famous "paradox": assuming that the probability of having one's birthday on any given day of the year is $\frac{1}{365}$ (which does not happen to be true, but, hey...), how many people should you gather at random so that the probability that at least two of them share their birthday is greater than $\frac{1}{2}$? The answer is surprisingly low. In fact, the probability that no two out of n share their birthday can be calculated as follows (of course, we need $n \leq 365$): looking at the n people in sequence, the first can have 365 possible birthdays, but the second only 364 (otherwise they would share their birthday), the third 363, and so on, down to $365-(n-1)$. On the other hand, if each individual can have any birthday, the total number of possible arrangements is 365^n. Hence the probability that no two share the same birthday is

$$p(n) = \frac{365!}{(365-n)!365^n}$$

By programming this expression and checking for a few values, it is easy to see that $p(n)$ decreases as n increases (that's pretty obvious), and that $p(22) =$

$.52, p(23) = .49$. In fact, $p(50) = .0296$, and so on, rapidly decreasing! As for our class, if all 35 "slots" were taken, plus your instructor, it would amount to 36 people, and, under this model, the probability of having no common birthdays is $p(36) = .1678$. See example 5i, on page 39 of the book.

As for the problem at hand, we have similarly that for none to share their birth month, they can have 12! ways of arranging them, out of a total of 12^{12}, yielding a probability of

$$\frac{12!}{12^{12}} \simeq 5.37 \cdot 10^{-5}$$

#50

At bridge, each of the four player (they form two teams at the table) is dealt 13 cards. There are plenty of possible hands, but we are looking at your hand and your partner's only. So, you can have $\binom{52}{13}$ possible hands, and, from the remaining 39 cards, your partner has $\binom{39}{13}$ possible deals.

Now, you hope to get 5 spades (out of 13), and 8 other cards (out of $52-13=$ 39 non-spade cards), while your partner is supposed to get all of the remaining 8 spades (out of 8, that is), and 5 other cards, from the $39-8=31$ remaining cards. All in all,

$$\frac{\binom{13}{5}\binom{39}{8}\binom{8}{8}\binom{31}{5}}{\binom{52}{13}\binom{39}{13}}$$

#54

The trick here is in the "at least": the number listed corresponds to the probability of having *exactly one* void suit. But we need to account for cases where we have 2 or 3 void suits as well. In other words, denoting by S, H, D, C the events that spades, hearts, diamonds or clubs are void, we have as probability for at least one void suit

$$P[S]+P[H]+P[D]+P[C]-P\left[S\bigcap H\right]-P\left[S\bigcap D\right]-P\left[S\bigcap C\right]-P\left[H\bigcap D\right]-$$

$$-P\left[H\bigcap C\right]-P\left[D\bigcap C\right]+P\left[S\bigcap H\bigcap D\right]+P\left[S\bigcap H\bigcap C\right]+P\left[S\bigcap D\bigcap C\right]+$$

$$+P\left[H\bigcap D\bigcap C\right]$$

By symmetry, events in each group (the first 4 terms, the next 6, the last 4) have the same probability.

The first four add up to the "wrong" answer. The second group of 6 adds up to

$$6\frac{\binom{26}{13}}{\binom{52}{13}}$$

and the third group of 4 adds up to

$$4\frac{\binom{13}{13}}{\binom{52}{13}}$$

for a total of

$$\frac{4\binom{39}{13}}{\binom{52}{13}} - \frac{6\binom{26}{13}}{\binom{52}{13}} + \frac{4}{\binom{52}{13}}$$