Solutions to the Problems in the First Assignment

Math 394 B & C

We use the more common notation $A \bigcap B$ for the intersection of the sets A and B (rather than AB, as in the book), and denote inclusion by $A \subseteq B$, rather than $A \subset B$, to distinguish between the possibility that $A = B$, and the exclusion of it (this is also the more common notation in mathematical literature). We will denote the enclosing set by Ω (again, following common practice), so that all sets considered below are subsets of Ω, and, in particular, $E^c = \Omega \setminus E$

1

One way to prove the statement is to calculate

$$E \bigcap \left(E \bigcap F\right) = \left(E \bigcap E\right) \bigcap F = E \bigcap F$$

hence $E \bigcap F \subseteq E$, and

$$E \bigcup \left(E \bigcup F\right) = \left(E \bigcup E\right) \bigcup F = E \bigcup F$$

so that $E \subseteq E \bigcup F$

2

If $E \subseteq F$, then $E \bigcap F = E$, so that $(E \bigcap F)^c = E^c \bigcup F^c = E^c$, which proves that $F^c \subseteq E^c$

3

Note that $E \bigcup E^c = \Omega$, hence, $(F \bigcap E) \bigcup (F \bigcap E^c) = F \bigcap (E \bigcup E^c) = F \bigcap \Omega = F$. The other relation is proved exactly in the same way.

4

The delicate point is in the precise definition of $\bigcup_{i=1}^{\infty} E_i$, and $\bigcap_{i=1}^{\infty} E_i$. The first is the smallest subset of Ω containing all $\bigcup_{i=1}^{n} E_i$, $i = 1, 2, \ldots$. Since, by induction,

$$F \bigcap \left(\bigcup_{i=1}^{n} E_i \right) = \bigcup_{i=1}^{n} \left(F \bigcap E_i \right)$$

it is easy to see that any $\omega \in \bigcup_{i=1}^{n} (F \bigcap E_i)$ will belong to $\bigcup_{i=1}^{n} E_i$, from which the statement follows. The second statement is proved in the same way, exchanging "smallest set containing all finite unions" with "largest set containing all finite intersections".

5

We can define

$$F_1 = E_1, \qquad F_i = E_i \setminus \bigcup_{k=1}^{i-1} F_k \quad i = 2, 3, \ldots$$

Now, any union can be written as the union of disjoint sets. Of course, $P[F_i] \leq P[E_i]$. Hence, we have

$$P\left[\bigcup_{i=1}^{\infty} E_i\right] = P\left[\bigcup_{i=1}^{\infty} F_i\right] = \sum_{i=1}^{\infty} P[F_i] \leq \sum_{i=1}^{\infty} P[E_i]$$

That is, we can prove, from the countable additivity of the probabilities of disjoint sets, the countable *subadditivity* of the probability of sets.

The idea can be pushed further. For example, define

$$F_1 = E_1, \quad F_i = E_i \bigcup F_{i-1}$$

Now, the sequence $\{F_i\}$ is increasing, since $F_1 \subseteq F_2 \subseteq \ldots$, so that any union can be written as the limit of an increasing sequence of sets. A similar argument shows that any intersection can be written as the limit of a decreasing sequence of sets. It follows that we can use the property of "continuity as set function" for probabilities as alternate axioms, in place of countable additivity. In fact, if you look at the proof of Proposition 6.1 in the book, you see that we only need continuity over increasing, or over decreasing sequences, to prove the other, and consequently, countable additivity.

7

(a) We can use the distributive property to conclude

$$\left(E\bigcup F\right)\bigcap\left(E\bigcup F^c\right) = E\bigcap\left(F\bigcup F^c\right) = E\bigcap\Omega = E$$

(b) We can use now the associative and commutative properties, together with point (a), and conclude, after applying again the distributive property at the end,

$$\left(E\bigcup F\right)\bigcap\left(E^c\bigcup F\right)\bigcap\left(E\bigcup F^c\right) = E\bigcap\left(E^c\bigcup F\right) = \left(E\bigcap E^c\right)\bigcup\left(E\bigcap F\right) = E\bigcap F$$

(c) Similarly,

$$\left(E\bigcup F\right)\bigcap\left(F\bigcup G\right) = F\bigcup\left(E\bigcap G\right)$$

(for example, $(E\bigcup F)\bigcap(F\bigcup G) = (F\bigcap[E\bigcup F])\bigcup(G\bigcap[E\bigcup F])$, and $(F\bigcap[E\bigcup F]) = F$, while $(G\bigcap[E\bigcup F]) = (G\bigcap E)\bigcup(G\bigcap F)$, so we end up with $F\bigcup(G\bigcap E)\bigcup(G\bigcap F)$, and since $F\bigcup(G\bigcap F) = F$, we hav the result)

9

Since we are dealing with a finite number of experiments (n), we only need to check finite unions. We obviously have

$$0 \leq \frac{n(E)}{n} \leq 1$$

and, if E and F are disjoint, they will never occur together, so that $n\left(E\bigcup F\right) = n\left(E\right) + n\left(F\right)$, from which additivity follows immediately. Of course, $n\left(S\right) = n$, which covers the last of the three axioms.

10

The easy way is to use Venn diagrams, with the intuition that "probabilities" assign a "weight" to any portion of a set. The idea is, of course, to "count" how many times we are "counting" parts the belong to two or three of the sets.

11

Note that $P\left[E\bigcup F\right] = P\left[E\right] + P\left[F\right] - P\left[E\bigcap F\right] \leq 1$, so, indeed,

$$P\left[E\bigcap F\right] \geq P\left[E\right] + P\left[F\right] - 1$$

12

In logic terms, this situation corresponds to “XOR”, “exclusive or” (either one or the other, but not both). Since $E \bigcap F$ is a subset of both E and F, we need to subtract its probability from both.

13

One way to prove this is to note that

$$E = E \bigcap \left(F \bigcup F^c \right) = \left(E \bigcap F \right) \bigcup \left(E \bigcap F^c \right)$$

where the union is of disjoint sets (since $F \bigcap F^c = \emptyset$). Hence,

$$P\left[E\right] = P\left[E \bigcap F\right] + P\left[E \bigcap F^c\right]$$