Sample Test #1

The test will include 3 quizzes, examples of which you can look up in the additional examples file under "Week 4" in the materials page. It will also include three word problems. While definitely not the same as the following ones, they should require essentially the same tools.

1

We throw a fair six-sided die. Let D_k be the event that the die turns up k ($k = 1, 2, \ldots, 6$). Depending on the result of the toss, if we observe D_j (that is, the die comes up with j points), we flip j fair coins. Let E_i be the event that the number of heads that result from the flips is i (for example, if D_3 occurred, and the flips result in HHT, the corresponding event is E_2).

1. Write a formula for $P\left[E_i \,|D_j\,\right]$. For which values of i and j is this conditional probability equal to zero?

2. Calculate $P\left[E_4\right]$

3. Calculate $P\left[D_5 \,|E_4\,\right]$

Solution:

1. We are tossing j coins, k of which are to be heads. We clearly need $1 \le j \le 6$, and $0 \le k \le j$. The probability of k heads out of j tosses is

$$\binom{j}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{j-k} = \binom{j}{k}\left(\frac{1}{2}\right)^j$$

(since the coins are fair).

2. By the total probability formula,

$$P\left[E = k\right] = \sum_{j=1}^{6} P\left[E = k \,|D = j\,\right] P\left[D = j\right] = \sum_{j=1}^{6}\binom{j}{k}\left(\frac{1}{2}\right)^j \frac{1}{6}$$

(where we agree that $\binom{j}{k} = 0$ when $k > j$). For $k = 4$,

$$P\left[E = 4\right] = \sum_{j=4}^{6}\binom{j}{4}\left(\frac{1}{2}\right)^j \frac{1}{6} = \frac{1}{6}\left(\frac{1}{2^4} + 5\frac{1}{2^5} + \frac{6 \cdot 5}{2}\frac{1}{2^6}\right) = \frac{1}{6 \cdot 16}\left(1 + \frac{5}{2} + \frac{15}{4}\right) =$$

$$= 0.075521$$

3. Using Bayes' Rule,

$$P\left[D = 5 \,|E = 4\,\right] = \frac{P\left[E = 4 \,|D = 5\,\right] P\left[D = 5\right]}{P\left[E = 4\right]} = \frac{5 \cdot \frac{1}{2^5} \cdot \frac{1}{6}}{0.075521} = .34483$$

2

Consider a transmission channel sending bits from a source to a destination. We are not sure about the noisiness of the channel, so that we try to check it and choose one of two models:

1. bits arrive correctly with probability .95, independently of each other
2. 1's arrive correctly with probability .90 and 0's with probability .98, independently of each other.

At first, we have no reason to prefer one model to the other, so we assign a priori probabilities 0.5 to each.

We receive a 1, then a 0, then a 1. The first two turn out to be correct, the third not. What is your assessment of the likelihood of the two models now?

Solution: Call the events that model 1 or 2 are correct, M_1 and M_2, respectively. Call C_i the event that the i-th bit is correctly transmitted.

$$P\left[C_1 \bigcap C_2 \bigcap C_3^c \,|M_1\right] = P\left[C_1\,|M_1\right] P\left[C_2\,|M_1\right] P\left[C_3^c\,|M_1\right] = .95{\cdot}.95{\cdot}.05 \approx .045$$

while

$$P\left[C_1 \bigcap C_2 \bigcap C_3^c \,|M_2\right] = P\left[C_1\,|M_2\right] P\left[C_2\,|M_2\right] P\left[C_3^c\,|M_2\right] = .90{\cdot}.98{\cdot}.02 \approx .018$$

Hence,

$$P\left[C_1 \bigcap C_2 \bigcap C_3^c\right] = P\left[C_1 \bigcap C_2 \bigcap C_3^c \,|M_1\right] P\left[M_1\right] + P\left[C_1 \bigcap C_2 \bigcap C_3^c \,|M_2\right] P\left[M_2\right] \approx 0.045{\cdot}0.5 + 0.018{\cdot}0.5 =$$

Bayes' theorem now results in

$$P\left[M_1 \left| C_1 \bigcap C_2 \bigcap C_3^c\right.\right] = \frac{P\left[C_1 \bigcap C_2 \bigcap C_3^c\,|M_1\right] P\left[M_1\right]}{P\left[C_1 \bigcap C_2 \bigcap C_3^c\right]} \approx \frac{0.045 \cdot 0.5}{0.0315} \approx 0.71$$

$$P\left[M_2 \left| C_1 \bigcap C_2 \bigcap C_3^c\right.\right] = \frac{P\left[C_1 \bigcap C_2 \bigcap C_3^c\,|M_2\right] P\left[M_2\right]}{P\left[C_1 \bigcap C_2 \bigcap C_3^c\right]} \approx \frac{0.018 \cdot 0.5}{0.0315} \approx 0.29$$

3

In a sequence of games of chance, each game results in a win with probability p, and all games are independent of each other. Suppose you win \$1 for each win, and lose \$3 for each loss. What is the probability that you will end up even or ahead after six games?

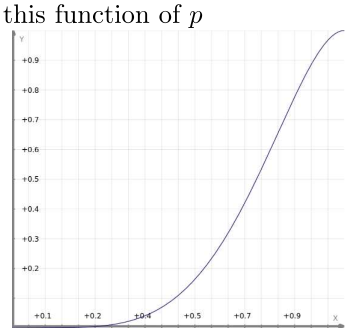
Solution: you need to win at least 5 games to be ahead. The probability of winning 5 is

$$\binom{6}{5} p^5 (1-p) = 6p^5 (1-p)$$

and the probability of winning 6 games is p^6. Hence, your probability is

$$6p^5 - 5p^6$$

Of course, you are really ahead with 5 or more wins – by at least \$2. A graph of this function of p



shows that you need close to $p = .75$ to have an even chance (if $p = 0.75$, your chance of being ahead is, approximately, 53%). After we introduce the notion of *expected value*, we will be able to get back to examples like this and discuss what, in our mind, might be the best definition of a "fair game".