Midterm #2 Solutions to the Problems

Math/Stat 394B

1 Quizzes

1.1

In the following list, mark the statements that are possible, leaving those that are impossible blank.

⊗ $EX = 1, EX^2 = 2$ There is nothing preventing a case like this

◯ $EX^2 = 1$, EX does not exist. We can see how this can't happen in a number of ways.

1. We have the inequality $(EX)^2 \leq E\left[X^2\right]$ that indicates that the first term cannot diverge if the second doesn't.

2. Thinking in terms of convergence of the integrals (in the absolutely continuous case), or the sums (in the discrete case), we see that convergence of $E\left[X^2\right]$ means, e.g. for the discrete case, that

$$\sum_k x_k^2 p_k < \infty \tag{1}$$

   Now, x_k has to be diverging for EX not to exist – otherwise all moments exist and we don't have a problem in the first place. Thus, eventually, $|x_k| \leq x_k^2$, and

$$|x_k|\, p_k \leq x_k^2 p_k$$

   and by the comparison theorem, the expected value $\sum_k x_k p_k$ is absolutely convergent.

3. It's OK to argue like in point 2 in a more informal way. For example, to say that since (in the absolutely continuous case) $\int_{-\infty}^{\infty} x^2 f_X(x)dx$ is convergent, we will have, asymptotically, something like

$$x^2 f_X(x) \leq \frac{1}{x^{1+\varepsilon}} \tag{2}$$

   for some $\varepsilon > 0$ (this is not quite rigorous, but it's close). If (2) holds, then, of course,

$$|x|\, f_X(x) \leq \frac{1}{x^{2+\varepsilon}}$$

   and $\int_{-\infty}^{\infty} x f_X(x)dx$ converges even better.

○ $EX = 2, EX^2 = 2$ The inequality in point 1 above shows this is impossible, since $2^2 = 4 > 2$

○ $EX = -1, Var[X] = -2$ Variances are non negative

1.2

In the following list, mark the statements that are *always* true.

○ $E[aX^2] = a^2 (E[X])^2$

⊗ $E[aX + bY] = aE[X] + bE[Y]$ This is the linearity of expectation.

○ $E[e^X] = e^{E[X]}$

○ $E[\log(XY)] = \log(E[X]) + \log(E[Y])$

○ $E[\max\{X, Y\}] = \max\{E[X], E[Y]\}$

All remaining relations are never true, except in the degenerate case, when the variables are actually *constants* (i.e., they have variance zero, i.e., they are equal to a number with probability one), so that $X = EX$, $E[X^2] = X^2$, $EY = Y$, $E[f(X, Y)] = f(X, Y)$ for any f.

1.3

Let X be a continuous RV, with a density $f_X(x)$ that is equal to zero outside the interval $[-10, 10]$, and positive inside. Let F_X be the corresponding cdf. In the following list mark the statements that are *certainly true*.

○ $F_X(0) = 0$. Since $f_X(x) > 0$ for $-10 < x < 0$, we'll have $F_X(0) = \int_{-10}^{0} f_X(x)dx > 0$

⊗ $F_X(-10) = 0$. We have been told that $P[X \leq 10] = 0$

○ $F_X(10) = 0$. Since $f_X(x) > 0$ for $-10 < x < 10$, $F_X(10) = \int_{-10}^{10} f_X(x)dx > 0$

○ $F_X(0) = 1$. Since $f_X(x) > 0$ for $0 < x < 10$, we'll have $F_X(0) = \int_{-10}^{0} f_X(x)dx < 1 = F_X(0) + \int_{0}^{10} f_X(x)dx$

○ $F_X(-10) = 1$. We already showed that $F_X(-10) = 0$

⊗ $F_X(10) = 1$. Since $f_X = 0$ for $x > 10$, we have that $\int_{-\infty}^{10} f_X(x)dx = F_X(10) = 1$.

2 Problems

2.1

You play repeatedly a lottery, where your chance of winning any given round is 10^{-6} (in lotteries, we normally assume that all rounds are independent)

1. What is the probability of winning *at least once* in 1000 attempts?
2. If the cost of a ticket is \$1, and the jackpot is \$500,000, what is your expected payoff for playing?
3. What is the pmf (probability mass function) of the number of wins over 2,000,000 attempts?
4. If you have a calculator, provide actual values of the pmf for the first few values (e.g., three). If you don't, write a usable formula (binomial coefficients would not be very usable here).

Solution: The number of wins over n attempts will be a binomial RV, with parameters $n, 10^{-6}$. If n is large, it will be well approximated by a Poisson RV with parameter $n \cdot 10^{-6}$

1. The probability of "at least once" is best computed as 1 minus the probability of "nothing". Hence, if W is the number of wins,

$$P[W>0] = 1 - P[W=0] = 1-(1-10^{-6})^{1000} \approx 1 - e^{10^3 \cdot 10^{-6}} = 1 - e^{-10^{-3}} \approx 9.995 \cdot 10^{-4}$$

or .1%. You get the same approximation by stopping Newton's binomial formula after the second term:

$$\left(1-10^{-6}\right)^{1000} = 1 - 1000 \cdot 1^{999} \cdot 10^{-6} + \ldots$$

2. For a Bernoulli RV with parameter p, the expected value is p. Hence, your expected payoff is $10^{-6} \cdot 5 \cdot 10^5 = 5 \cdot 10^{-1}$. Since you pay \$1 to play, you will, on average, lose \$0.50. In real life, of course, the odds are much worse than these.
3. Over 2,000,000 attempts, we would have the distribution $bin\left(2 \cdot 10^6, 10^{-6}\right)$, or

$$P[W=k] = \binom{2 \cdot 10^6}{k} 10^{-6k}\left(1-10^{-6}\right)^{2 \cdot 10^6 - k}$$

With an expected value of $2 \cdot 10^6 \cdot 10^{-6} = 2$, and a variance of $2 \cdot 10^6 \cdot 10^{-6}\left(1-10^{-6}\right) \simeq 2$, we can reasonably use the Poisson approximation, with $\lambda = 2$, and compute

$$P[W=0] = e^{-2} = .13534$$

$$P[W=1] = 2e^{-2} = .27067$$

$$P[W=2] = \frac{4}{2}e^{-2} = 2e^{-2} = .27067$$

$$P[W=3] = \frac{2^3}{6}e^{-2} = \frac{4}{3}e^{-2} = .18045$$

$$P[W=4] = \frac{2^4}{24}e^{-2} = \frac{2}{3}e^{-2} = .090224$$

$$\ldots$$

2.2

A system is composed of n components, each having a random time to failure T_i, $i = 1, 2, \ldots, n$ (that is, component i will break down at time T_i). Assume all components fail independently of each other, and have exponential distributions with parameter λ_i.

1. How likely is it for component $\#i$ to last longer than its *expected* failure time?
2. If the components are arranged as a *series system* (that is, they have to be all functioning for the system as a whole to function), what is the MTTF (*Mean Time To Failure*) for the system as a whole?
3. If the components are arranged as a *stand-by system* (they are used one at a time, with the next component taking the place of the previous as soon as that breaks down), what is the MTTF of the system?
4. Assume that n is very large, and all parameters are equal, $\lambda_i = \lambda$, $i = 1, 2 \ldots, n$, sufficiently small to justify the use of a Gaussian approximation for the system failure time T in the stand-by configuration. Use this approximation to find the (approximate) size of the largest fluctuation around $E[T]$ occurring with a probability of 95% (in other words, find h such that $P[|T - E[T]| \leq h] = 0.95$)

Solution:

1. We are asking what is
$$P\left[T > \frac{1}{\lambda}\right]$$
when T is an exponential with parameter λ. Since
$$P[T > t] = e^{-\lambda t}$$
$$P\left[T > \frac{1}{\lambda}\right] = e^{-1} = .36788$$

2. If all components need to be working, the comprehensive failure time is $T = \min\{T_1, T_2, \ldots, T_n\}$. Since, by an argument we have seen a number of times, $R_T = R_{T_1} \cdot R_{T_2} \cdot \ldots R_{T_n}$, T will be an exponential with parameter $\sum_{k=1}^{n} \lambda_k$, and its expected value (the MTTF of the system) will be $\frac{1}{\sum_{k=1}^{n} \lambda_k}$

Here, $R_X(x) = P[X > x] = 1 - F_X(x)$ is the "survival function", also denoted by $\overline{F}_X$.

3. In the standby mode, the comprehensive time to failure is just the sum of the failure times. Hence, since $T = \sum_{k=1}^{n} T_k$,
$$ET = \sum_{k=1}^{n} \frac{1}{\lambda_k}$$

4. If n is very large, and the failure times are independent, and identically distributed,
$$ET = \frac{n}{\lambda}, \quad Var\,[T] = \frac{n}{\lambda^2}$$
hence
$$\frac{T - ET}{\sqrt{Var\,[T]}} = \frac{T - \frac{n}{\lambda}}{\frac{\sqrt{n}}{\lambda}}$$
is, approximately, a standard normal. Hence, with a probability of approximately 95%,
$$-\frac{1.96\sqrt{n}}{\lambda} < T - \frac{n}{\lambda} < \frac{1.96\sqrt{n}}{\lambda}$$

provided, of course, that λ is also very large to make the CLT a reasonable tool (we need to add "many small" items, where "small" is evaluated here from the size of $\frac{1}{\lambda}$, which is the value of the standard deviation. If you look at the statement (and the proof) of the CLT,t you will notice that you are looking for the standard deviation of the individual terms in the sum to be of the order of $\frac{1}{\sqrt{n}}$ – in our case, we want λ^2 to be of the order of n.