Midterm #1 - Solutions

Math/Stat 394

1 Quizzes

1.1

Let A be the event "Andrea and Bill are both in class". The *complementary event* is (choose one):

○ A^c =”*Neither* Andrea *nor* Bill are in class”

○ A^c =”Bill is not in class”

⊗ A^c =”*At least* one of the two is not in class”

○ A^c =”Andrea is not in class”

Discussion: If it is not true that *both* are in class, then at least one must not be there. We can also formalize the statement: let C_A =”Andrea is in class”, and C_B =”Bill is in class”. $A = C_A \bigcap C_B$, so that

$$A^c = \left(C_A \bigcap C_B\right)^c = C_A^c \bigcup C_B^c$$

i.e., either on or both are not in class.

1.2

Which of the following statements would imply that two events, A, B, with $P[A] > 0, P[B] > 0$ are such that $P[A|B] = P[B|A]$?

○ A and B are independent

○ $B \subseteq A$

○ $P[A \bigcup B] = 2P[A \bigcap B]$

⊗ $P[A] = P[B]$

Discussion: We need $\frac{P[A \bigcap B]}{P[B]} = \frac{P[A \bigcap B]}{P[A]}$.

1.3

Suppose we have two events A, B, and $P[A] > 0, P[B] > 0$. Which of the following is *always* true?

◯ $P[A|B] \neq P[B|A]$

⊗ $P[A|B] = 1 - P[A^c|B]$

◯ $P[A|B] = 1 - P[A|B^c]$

◯ $P[A|B] = P[A]$ if, additionally, $A \bigcap B = \emptyset$

Discussion: For fixed B, $P[\cdot|B]$ is a probability. Hence, it satisfies all the axioms and basic properties of probabilities, including the fact that the probability of an event is equal to $1-$ the probability of its complement. As for the other option, the first one is often true, but not always (its opposite requires $P[A] = P[B]$, due to Bayes' Formula). The third is unlikely, as conditioning to an event or to its complement are unrelated operations. The last is always false (it means A and B are independent *and disjoint, at the same time*), unless $P[A] = 0$, that is in a trivial case.

2 Problems

2.1

We toss two independent "fair" dice. Let $X_i : i = 1, 2, \ldots 6$, and $Y_j : j = 1, 2, \ldots, 6$ be, respectively, the events such that the first die comes out with i points, and the second die with j points. Let $Z_k : k = 2, 3, \ldots, 12$ be the event that the sum of the two is equal to k. Compute

1. $P[Z_5|X_2]$
2. $P[X_2|Z_5]$

Solutions:

1. If the first die turns out 2, the only way for the sum to be 5 is if the second die turns 3, which has probability $\frac{1}{6}$. More formally,

$$P[Z_5|X_2] = \frac{P[Z_5 \bigcap X_2]}{P[X_2]} = \frac{P[X_2 \bigcap Y_3]}{P[X_2]} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

2. We can use Bayes' Rule now:

$$P[X_2|Z_5] = \frac{P[Z_5|X_2]\,P[X_2]}{P[Z_5]} = \frac{\frac{1}{6} \cdot \frac{1}{6}}{\frac{4}{36}} = \frac{1}{4}$$

   We can also argue directly: if the sum is 5, X can be either 1, 2, 3, or 4, each outcome having the same probability.

2.2

We throw a fair six-sided die. Let D_k be the event that the die turns up k ($k = 1, 2, \ldots, 6$). Depending on the result of the toss, if we observe D_j (that is, the die comes up with j points), we flip j fair coins. Let E_i be the event that the number of heads that result from the flips is i (for example, if D_3 occurred, and the flips result in HHT, the corresponding event is E_2).

1. Write a formula for $P[E_i \,|D_j\,]$. For which values of i and j is this conditional probability equal to zero?
2. Calculate $P[E_4]$
3. Calculate $P[D_5 \,|E_4\,]$

Solution:

1. We are tossing j coins, k of which are to be heads. We clearly need $1 \le j \le 6$, and $0 \le k \le j$. The probability of k heads out of j tosses is

$$\binom{j}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{j-k} = \binom{j}{k}\left(\frac{1}{2}\right)^j$$

(since the coins are fair).

2. By the total probability formula,

$$P[E=k] = \sum_{j=1}^{6} P[E=k\,|D=j\,]\,P[D=j] = \sum_{j=1}^{6}\binom{j}{k}\left(\frac{1}{2}\right)^j\frac{1}{6}$$

(where we agree that $\binom{j}{k} = 0$ when $k > j$). For $k = 4$,

$$P[E=4] = \sum_{j=1}^{6}\binom{j}{4}\left(\frac{1}{2}\right)^j\frac{1}{6} = \frac{1}{6}\left(\frac{1}{2^4} + 5\frac{1}{2^5} + \frac{6\cdot5}{2}\frac{1}{2^6}\right) = \frac{1}{6\cdot16}\left(1 + \frac{5}{2} + \frac{15}{4}\right) =$$

$$= 0.075521$$

3. Using Bayes' Rule,

$$P[D=5\,|E=4] = \frac{P[E=4\,|D=5]\,P[D=5]}{P[E=4]} = \frac{5\cdot\frac{1}{2^5}\cdot\frac{1}{6}}{0.075521} = .34483$$

2.3

A transmission channel delivers bits (0s and 1s) at a rate of 60% 0s and 40% 1s. Suppose that, due to noise, a 0 is transmitted as a 1 5%, and a 1 is transmitted as a 0 3% of the time.

1. What is the probability that a given bit is correct?
2. You receive two bits as 11. What is the probability of both being correct if bits arrive independently
3. *A "parity check" bit is added to every pair of bits: this third bit is a 1 if the two bits are different, and 0 if their equal. This bit is obviously subject to the same noise as the others. A triplet is received as 110. What is the probability that the two meaningful bits were indeed 11? Compare with the same question when we receive a 11, no bit parity is added.

Solutions:

1. Let C be the event that the bit arrived correctly, and S_0, and S_1 the events that the transmitted bit was, respectively, a 0 or a 1. By a familiar formula

$$P[C] = P[C|S_0]P[S_0] + P[C|S_1]P[S_1] = .95 \cdot .6 + .97 \cdot .4 = .958$$

2. Assuming independence of the two bits, If no parity bit is added, and we receive a 11, the probability that the transmission is correct is

$$P\left[S_1^1 \bigcap S_1^2 \middle| R_1^1 \bigcap R_1^2\right] = \frac{P\left[S_1^1 \bigcap R_1^1 \bigcap S_1^2 \bigcap R_1^2\right]}{P\left[R_1^1 \bigcap R_1^2\right]}$$

Both the numerator and denominator can be factored, since the two bits are independent:

$$P\left[S_1^1 \bigcap R_1^1 \bigcap S_1^2 \bigcap R_1^2\right] = P\left[S_1^1 \bigcap R_1^1\right] P\left[S_1^2 \bigcap R_1^2\right] \qquad P\left[R_1^1 \bigcap R_1^2\right] = P\left[R_1^1\right] P\left[R_1^2\right]$$

We also have

$$\frac{P\left[S_1^1 \bigcap R_1^1\right]}{P\left[R_1^1\right]} = \frac{P\left[R_1^1 | S_1^1\right] P\left[S_1^1\right]}{P\left[R_1^1 | S_1^1\right] P\left[S_1^1\right] + P\left[R_1^1 | S_0^1\right] P\left[S_0^1\right]} =$$

$$= \frac{0.97 \cdot 0.4}{0.97 \cdot 0.4 + 0.05 \cdot 0.6} = \frac{.388}{.418} \approx .928$$

The same calculation holds for the second bit, so that we end up with the square of this number, approximately 0.86. As we can check from the last question, adding a parity bit enhances the reliability considerably.

3. Consider all possible events resulting in us receiving 110:

- 11 was transmitted, arrived correctly, and the parity bit arrived correctly as well.
- One of the two leading bits arrived wrong (this can happen in two ways), and the parity bit arrived wrong as well.
- Both main bits arrived wrong (a 00 was transmitted), and the parity bit arrived correctly.

We are obviously assuming that all bits arrive correctly or not independently from each other. Let's call S_i^j the event that the $j-$th bit transmitted was i, and R_i^j the event that the $j-$th bit was received as i. We are asking for

$$P\left[S_1^1 \bigcap S_1^2 \middle| R_1^1 \bigcap R_1^2 \bigcap R_0^3\right]$$

Note that the conditioned event, $S_1^1 \bigcap S_1^2$ is the same as $S_1^1 \bigcap S_1^2 \bigcap S_0^3$, since the first two bits determine the third. It is natural to apply Bayes' formula:

$$P\left[S_1^1 \bigcap S_1^2 \middle| R_1^1 \bigcap R_1^2 \bigcap R_0^3\right] = \frac{P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3 \middle| S_1^1 \bigcap S_1^2\right] P\left[S_1^1 \bigcap S_1^2\right]}{P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3\right]}$$

Looking at the three probabilities in the formula we can say

- $P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3 \middle| S_1^1 \bigcap S_1^2\right]$ is the probability that the two leading bits, 1's, arrived correctly, and, hence, that the parity bit, which must have been a 0, arrived correctly. Since bits travel independently of each other, that's equal to $.97 \cdot .97 \cdot .95 = 0.893855$.

- $P\left[S_1^1 \bigcap S_1^2\right]$ is the probability that two 1's were transmitted, which, again by independence, occurs with probability $.4 \cdot .4 = .16$.

- $P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3\right]$ is the probability of our observation to occur, and that is the result of one of the three events listed at the beginning to occur. By the usual formula, we have

$$P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3\right] = P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3 \middle| S_1^1 \bigcap S_1^2\right] P\left[S_1^1 \bigcap S_1^2\right] +$$

$$+P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3 \middle| S_1^1 \bigcap S_0^2\right] P\left[S_1^1 \bigcap S_0^2\right] +$$

$$+P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3 \middle| S_0^1 \bigcap S_1^2\right] P\left[S_0^1 \bigcap S_1^2\right] +$$

$$P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3 \middle| S_0^1 \bigcap S_1^2\right] P\left[S_0^1 \bigcap S_0^2\right]$$

Looking at each of the terms, we have

- $P\left[S_1^1 \bigcap S_1^2\right] = .4 \cdot .4 = .16$
- $P\left[S_0^1 \bigcap S_1^2\right] = P\left[S_1^1 \bigcap S_0^2\right] = .4 \cdot .6 = .24$
- $P\left[S_1^1 \bigcap S_1^2\right] = .6 \cdot .6 = .36$
- $P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3 \middle| S_1^1 \bigcap S_1^2\right]$ is the probability that the leading bits, both 1s, and the parity bit, a 0, all came in correctly. We already calculated this term:$.97 \cdot .97 \cdot .95 = 0.893855$
- $P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3 \middle| S_1^1 \bigcap S_0^2\right] = P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3 \middle| S_0^1 \bigcap S_1^2\right]$ is the probability that one of the two leading bits, which was a 0, came in corrupted, while the other was received correctly, and the parity bit, being a 1, was also corrupted. That's equal to $.97 \cdot .05 \cdot .03 = 0.001455$
- $P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3 \middle| S_0^1 \bigcap S_0^2\right]$ is the probability that both leading bits, 0s, arrived corrupted, and hence the parity bit was transmitted correctly. This probability is $.05 \cdot 0.05 \cdot 0.95 = 0.002375$

Collecting all the above,

$$P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3\right] = 0.893855 \cdot .16 + 2 \cdot 0.001455 \cdot .24 + 0.002375 \cdot .36 = 0.1445702$$

and we end up with

$$\frac{P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3 \left| S_1^1 \bigcap S_1^2 \right.\right] P\left[S_1^1 \bigcap S_1^2\right]}{P\left[R_1^1 \bigcap R_1^2 \bigcap R_0^3\right]} = \frac{0.893855 \cdot .16}{0.1445702} \approx 0.989255$$