# Laminar Flow of Two Immiscible Fluids Through a Square Duct

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## Introduction

**Problem Description:**

Consider two immiscible fluids flowing through a square channel. More than one possible flow arrangement can occur. The three possibilities studied in this research are shown schematically in Figures 1, 2, and 3. The flow arrangement shown in Figure 1 is referred to as the "Top-Bottom" configuration where the two fluids flow down the channel with one on top of the other. The flow arrangement shown in Figure 2 is referred to as the "Left-Right" configuration where the two fluids flow side-by-side down the channel. The flow arrangement shown in Figure 3 is referred to as the "Inside-Outside" configuration where Fluid 1 always surrounds Fluid 2. Fluid 2 has a viscosity that is 1.5 times greater than the viscosity of Fluid 1.

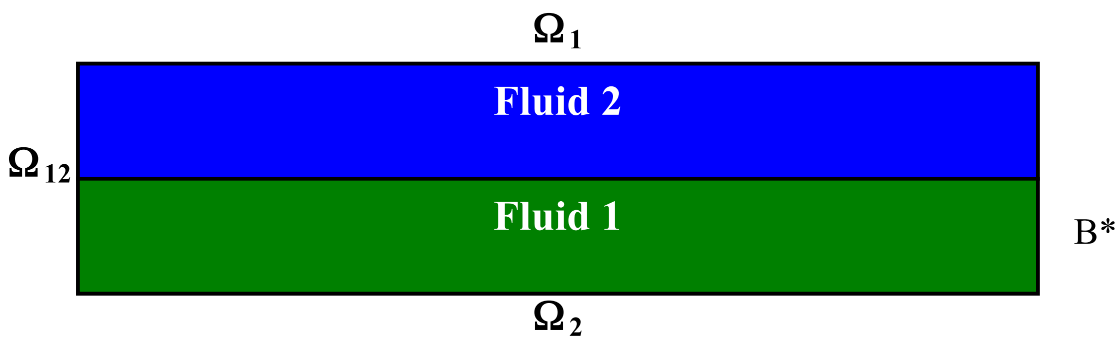


Figure 1. The Top-Bottom flow arrangement is shown. The velocity of the fluids is zero along $\Omega_1$ and $\Omega_2$ with a natural boundary condition along $\Omega_{12}$ that maintains the continuity of velocity. B* gives the location of the boundary and is measured starting from the bottom and going up. Thus, B* is proportional to the fraction of space occupied by Fluid 1.

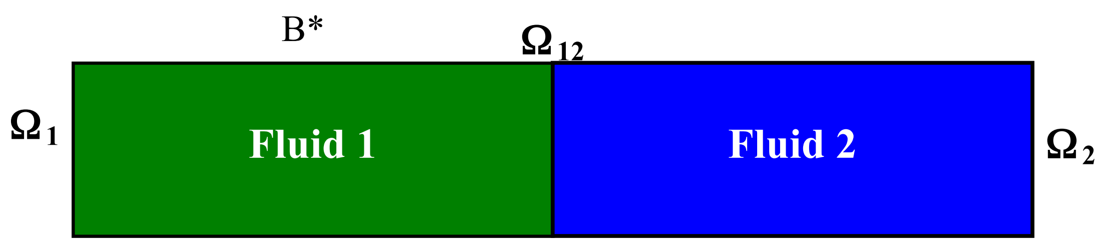


Figure 2. The Left-Right flow arrangement is shown. Again, the velocity of the fluids is zero along $\Omega_1$ and $\Omega_2$ with a natural boundary condition along $\Omega_{12}$ that maintains the continuity of velocity. B* gives the location of the boundary and is measured starting from the left and moving right. Thus, B* is proportional to the fraction of space occupied by Fluid 1.

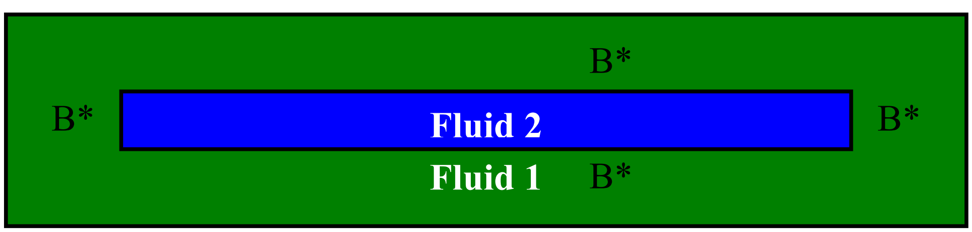


Figure 3. The Inside-Outside flow arrangement is shown. B* gives the location of the boundaries and is measured starting from the outside and moving inside. Thus, B* is proportional to the fraction of space occupied by Fluid 1. It is the same distance in all directions surrounding the domain of Fluid 2.

**Objectives:**
This experiment has three objectives. First, FEMLAB is used to model the situation where the two fluids have different viscosities. The boundary location between the two fluids is varied for each run.

Once the equation governing this situation is derived using variational principles,

$$\Phi(u_1,u_2) = \left[\frac{1}{2}\int_{\Omega_1} \mu_1 \nabla u_1 \bullet \nabla u_1 \, d\Omega \quad - \quad \frac{\Delta P}{L}\int_{\Omega_1} u_1 dxdy\right] + \left[\frac{1}{2}\int_{\Omega_2} \mu_2 \nabla u_2 \bullet \nabla u_2 \, d\Omega_2 \quad - \quad \frac{\Delta P}{L}\int_{\Omega_2} u_2 dxdy\right] \qquad \text{(Equ. 1)}$$

the second task is to use the Subdomain Integration command in FEMLAB to calculate the value of each term of the equation for each simulation. The overall value of the equation needs to be minimized as is discussed later.

Third, the flow rate of both fluids is plotted as a function of the fraction of space occupied by Fluid 1 to determine the boundary location where the fluid flow rates are identical

$$\int_{\Omega_1} u_1 dxdy = \int_{\Omega_2} u_2 dxdy \qquad \text{(Equ. 2)}$$

**Purpose:**
The purpose of this research is to determine the actual orientation that the two fluids take relative to one another under the circumstances described above.

**Theory:**
The term $\Phi$ in Equation 1 is called a functional. The domain of the functional consists of all admissible functions. A function is admissible if it satisfies the boundary conditions for the problem, is continuous, and has continuous first derivatives. An extremal is an admissible function that makes the variation, the derivative of the functional, equal to zero

$$\varepsilon \lim_{\varepsilon \to 0} \frac{\Phi(y+\varepsilon\eta) - \Phi(y)}{\varepsilon} = 0 \qquad \text{(Equ. 3)}$$

Therefore, the extremal minimizes (or maximizes) $\Phi$. The sign of the second derivative determines whether a minimum or maximum has been found.

For the case of laminar flow through ducts, the functional is proportional to the volumetric flow rate of each fluid[3]. For this problem, the admissible functions must satisfy zero velocity at the walls, and the extremals must satisfy Poisson's Equation. In addition, the negative of $\Phi$ is maximized here, which results in the minimization principle as indicated earlier.

**Limitations:**
Certain limitations apply to the simulations and any results obtained from them. First, the flow of both fluids is fully developed from beginning to end in the simulations. Thus, no entry effects are considered. Also, if there is any diffusion across the boundary between the two fluids, it does not change the viscosity of either fluid.

## Materials

The materials used in this research are FEMLAB version 2.3.0.148 and Microsoft Excel.

## Procedure

**Simulation Set-up and Solution:**
After opening Matlab, the command "FEMLAB" is issued in the Matlab command window. To find Poisson's Equation, one must select 2-Dimensional and then Classical PDE's. Then, double-click on Poisson's Equation

$$-\mu\nabla^2 u = \frac{\Delta P}{L} \qquad \text{(Equ. 4)}$$

Next, the rectangular duct shown in Figures 1 and 2 is created by drawing two separate rectangles next to each other. The rectangles are drawn so that the boundary between them is the location of the boundary between the two fluids. The gridline spacing can be changed to achieve any boundary location. The width of the channel is three, and the height of the channel is one.

For the situation shown in Figure 3, the domain of Fluid 1 is created by connecting four rectangles around the perimeter of the channel and then making them a Composite Object. Another rectangle is drawn in the space in the middle to represent the domain of Fluid 2.

Then, the boundary conditions are set. For this problem, the velocity is zero at the walls of the duct. Therefore, the BC's in FEMLAB are

$$h = 1$$
$$r = 0$$

These are the default Dirichlet boundary conditions in FEMLAB, so no actual changes need to be made. Next, the Subdomain settings are specified. The viscosity of Fluid 1 is always equal to one. The value of the pressure drop per unit length is also one. Therefore, for the domain of Fluid 1 (Subdomain 1 in FEMLAB), the BC's are

$$c = 1$$
$$f = 1$$

Again, these are the default settings, so no actual changes need to be made. For the domain of Fluid 2 (Subdomain 2 in FEMLAB), use

$$c = A$$
$$f = 1$$

where A is the viscosity ratio and is defined as

$$A = \frac{\mu_2}{\mu_1} \qquad \text{(Equ. 5)}$$

For all simulation results presented in this report, a viscosity of 1.5 is chosen for Fluid 2. Thus, A is set to 1.5 in all simulations when a parametric solution is not performed. Once the experimental set-up is complete, the mesh is initialized and refined twice. Finally, the Solve button is clicked once.

**Calculations:**
Equation 1 is broken into four terms and re-written as

$$\Phi(u_1, u_2) = \frac{1}{2}(Term1) - (Term2) + \frac{1}{2}(\mu_2)(Term3) - (\mu_2)(Term4) \qquad \text{(Equ. 6)}$$

FEMLAB can calculate Terms 1 through 4 of this equation using the following procedure. Click Post, then Subdomain Integration. Choose the Subdomain of interest, and type the expression for the computer to integrate. For Terms 1 and 3, type

$$u_x^2 + u_y^2$$

The subscripting is not necessary. For Terms 2 and 4, simply type

$$u$$

or choose it from the list of options. Then, each result is substituted into Equation 6 and the value of $\Phi$ is calculated. It should be noted that FEMLAB uses its own numbering system for the Subdomains regardless of the order in which the rectangles are drawn or how they are oriented on the screen. For the Left-Right and Top-Bottom configurations, Subdomain 1 is always the rectangle on the left or on the bottom. Subdomain 2 is always on the right or on top. Thus, Fluid 1 is always the fluid on the left or bottom, and Fluid 2 is always on the right or on top.

For the Inside-Outside configuration, the less viscous fluid (Fluid 1) is on the outside near the walls (Subdomains 1, 2, 4, and 5), and the more viscous fluid (Fluid 2) is flowing down the center of the channel (Subdomain 3). This situation is more probable physically than having the less viscous fluid flowing down the center because the highly viscous fluid would then have to flow near the stationary wall.

## Results

**Simulation Results:**
Sample velocity field plots for the Left-Right configuration are provided below.

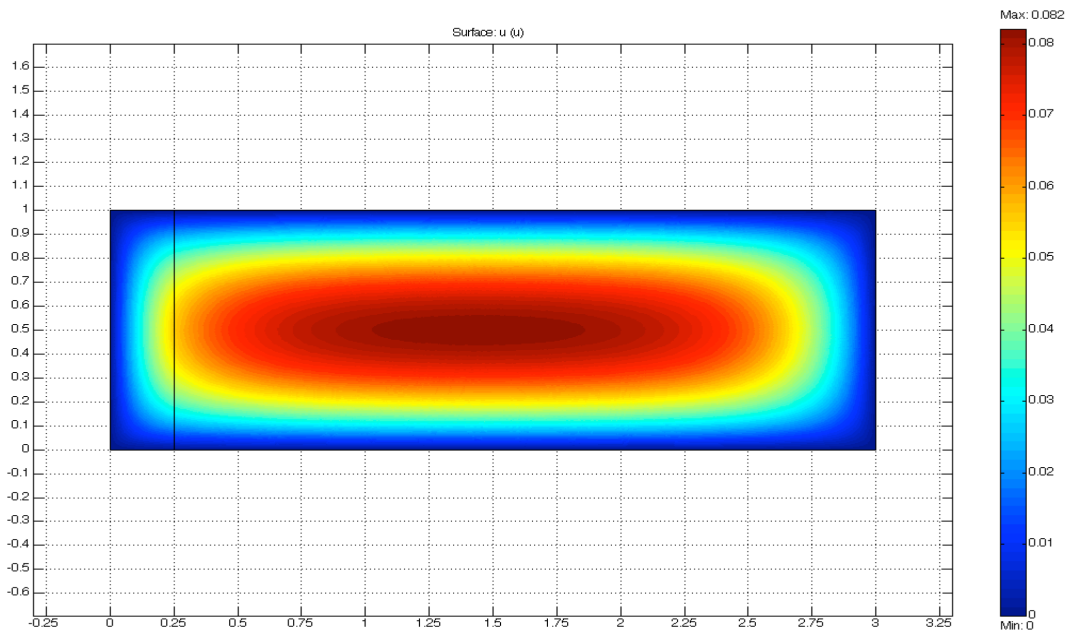


Figure 4. This plot represents the case where one-twelfth of the channel is taken up by Fluid 1 (B* = 0.25). Because the channel is nearly 92% Fluid 2, there is hardly any disruption in the velocity field.

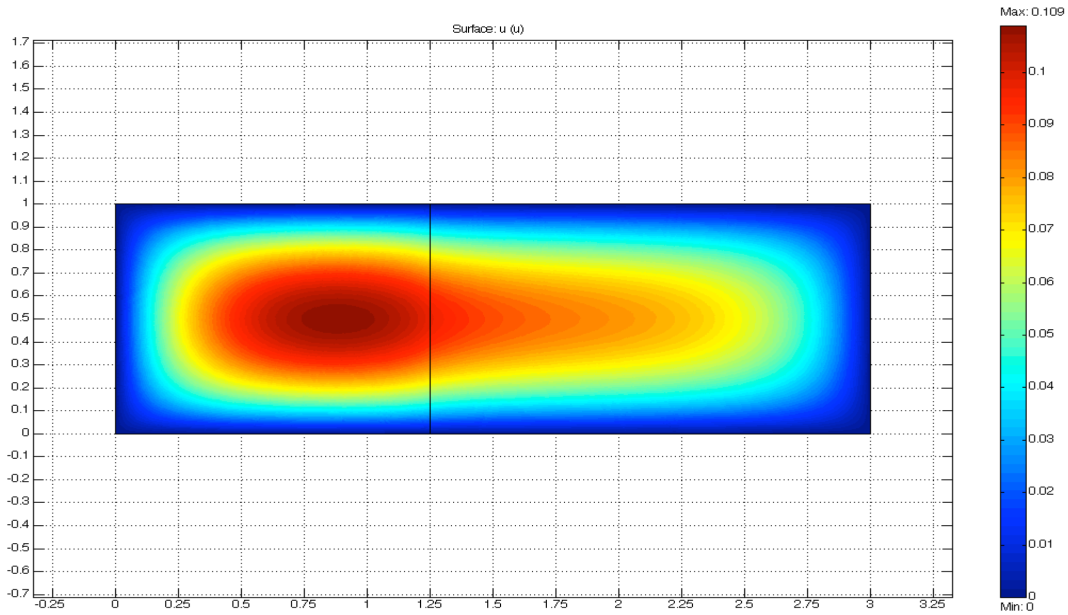


Figure 5. This plot represents the case where five-twelfths of the channel is taken up by Fluid 1 (B* = 1.25). The location of the boundary in this simulation is the closest to the boundary location that actually makes the two fluid flow rates equal. In this simulation, only 58.3% of the channel is Fluid 2.

Sample velocity field plots for the Top-Bottom configuration are provided below.

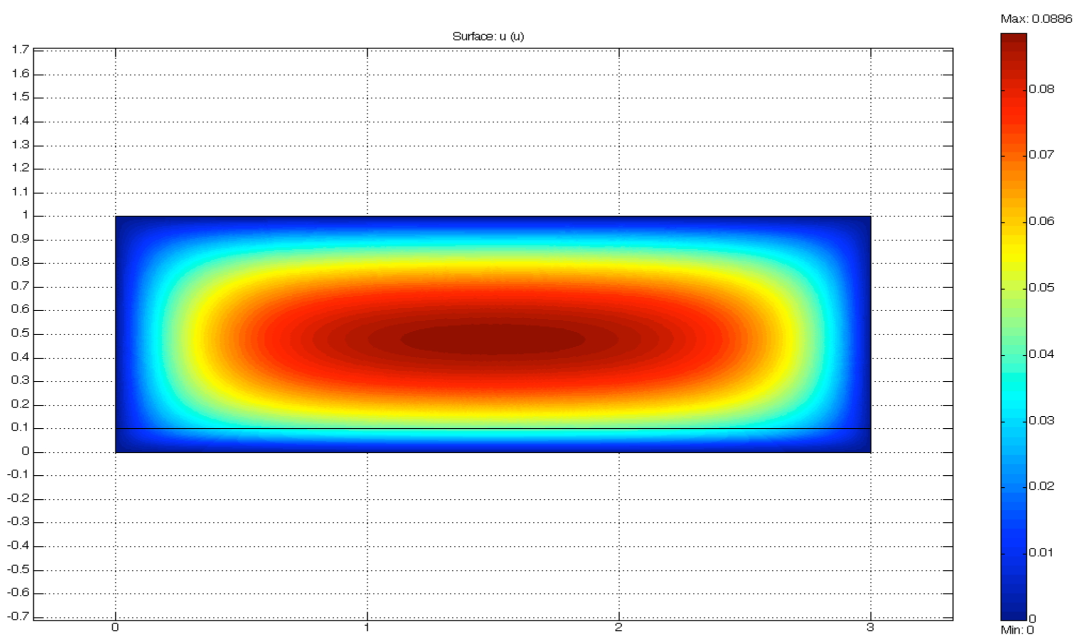


Figure 6. This plot represents the case where one-tenth of the channel is taken up by Fluid 1 (B* = 0.1). Because the channel is 90% Fluid 2, there is hardly any disruption in the velocity field.

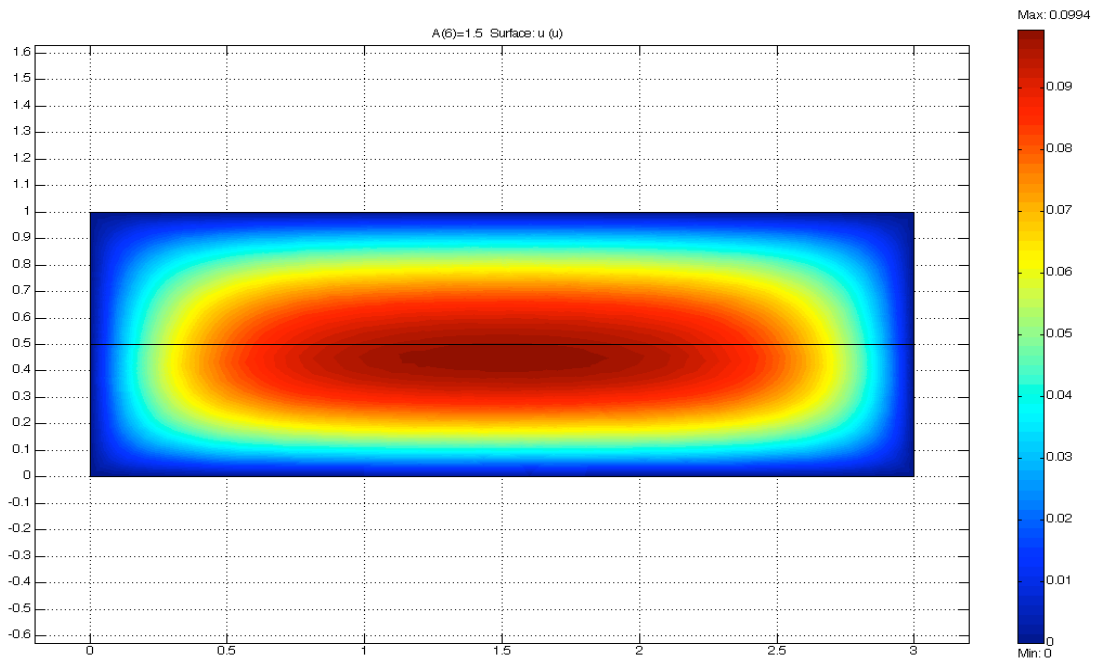


Figure 7. This plot represents the case where the channel is half-filled with each fluid (B* = 0.5). The maximum velocity does not occur along the centerline boundary. It occurs below it within the Fluid 1 Subdomain.

A sample velocity field plot for the Inside-Outside configuration is provided below.

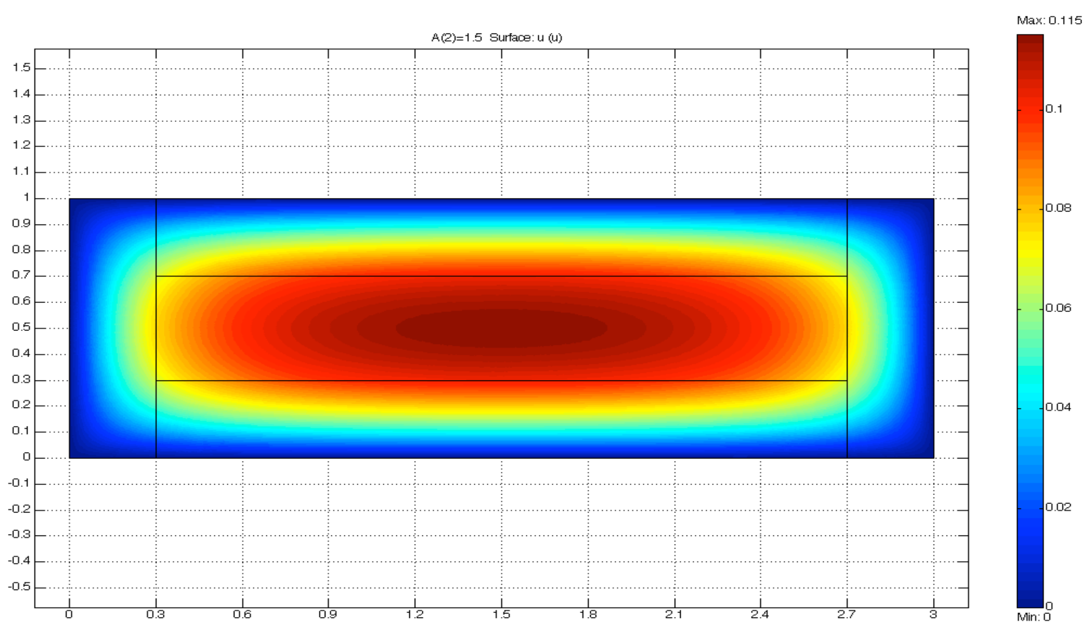


Figure 8. This plot represents the case where the channel is 32% Fluid 2 (B* = 0.3). The location of the boundary in this simulation is the closest to the boundary location that actually makes the two fluid flow rates equal.

**Numerical Results:**

The tabulated results of the Subdomain Integration are given below. B' is the fraction of space occupied by Fluid 1 and is found by dividing B* by the greatest value that B* can take in that particular direction and configuration (normalizing B*). All results pertain to the case where the viscosity ratio is equal to 1.5.

Table 1. Subdomain Integration results for the Left-Right configuration.

| B* | B' | Term 1 | Term 2 | Term 3 | Term 4 | Integral |
|---|---|---|---|---|---|---|
| 0.00 | 0.00 | 0 | 0 | 0.0878 | 0.1317 | -0.0658 |
| 0.25 | 0.08 | 0.0094 | 0.0056 | 0.0839 | 0.1297 | -0.0677 |
| 0.50 | 0.17 | 0.0189 | 0.0182 | 0.0800 | 0.1207 | -0.0694 |
| 0.75 | 0.25 | 0.0331 | 0.0348 | 0.0738 | 0.1091 | -0.0719 |
| 1.00 | 0.33 | 0.0505 | 0.0535 | 0.0662 | 0.0962 | -0.0749 |
| 1.25 | 0.42 | 0.0697 | 0.0734 | 0.0577 | 0.0828 | -0.0781 |
| 1.50 | 0.50 | 0.0896 | 0.0937 | 0.0488 | 0.0692 | -0.0814 |
| 1.75 | 0.58 | 0.1098 | 0.1141 | 0.0399 | 0.0555 | -0.0848 |
| 2.00 | 0.67 | 0.1297 | 0.1343 | 0.0310 | 0.0418 | -0.0881 |
| 2.25 | 0.75 | 0.1488 | 0.1540 | 0.0224 | 0.0284 | -0.0912 |
| 2.50 | 0.83 | 0.1662 | 0.1724 | 0.0145 | 0.0156 | -0.0940 |
| 2.75 | 0.92 | 0.1813 | 0.1877 | 0.0075 | 0.0049 | -0.0963 |
| 3.00 | 1.00 | 0.1975 | 0.1975 | 0 | 0 | -0.0987 |

Table 2. Subdomain Integration results for the Top-Bottom configuration.

| B* | B' | Term 1 | Term 2 | Term 3 | Term 4 | Integral |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0.0878 | 0.1317 | -0.0658 |
| 0.1 | 0.1 | 0.0381 | 0.0053 | 0.0714 | 0.1399 | -0.0726 |
| 0.2 | 0.2 | 0.0560 | 0.0188 | 0.0643 | 0.1337 | -0.0762 |
| 0.3 | 0.3 | 0.0641 | 0.0377 | 0.0614 | 0.1185 | -0.0781 |
| 0.4 | 0.4 | 0.0685 | 0.0602 | 0.0598 | 0.0981 | -0.0791 |
| 0.5 | 0.5 | 0.0732 | 0.0846 | 0.0577 | 0.0753 | -0.0799 |
| 0.6 | 0.6 | 0.0813 | 0.1096 | 0.0538 | 0.0525 | -0.0810 |
| 0.7 | 0.7 | 0.0954 | 0.1342 | 0.0470 | 0.0317 | -0.0829 |
| 0.8 | 0.8 | 0.1177 | 0.1573 | 0.0364 | 0.0150 | -0.0861 |
| 0.9 | 0.9 | 0.1510 | 0.1785 | 0.0210 | 0.0039 | -0.0912 |
| 1 | 1 | 0.1975 | 0.1975 | 0 | 0 | -0.0987 |

Table 3. Subdomain Integration results for the Inside-Outside configuration.

| B* | $Q_1$ | $Q_2$ | $\Phi$ |
|---|---|---|---|
| 0.2 | 0.0484 | 0.1351 | -0.1593 |
| 0.3 | 0.0958 | 0.0973 | -0.1452 |
| 0.4 | 0.1477 | 0.0490 | -0.0984 |

**Graphical Results:**
The following plots are obtained by graphing the data presented in Tables 1 and 2.

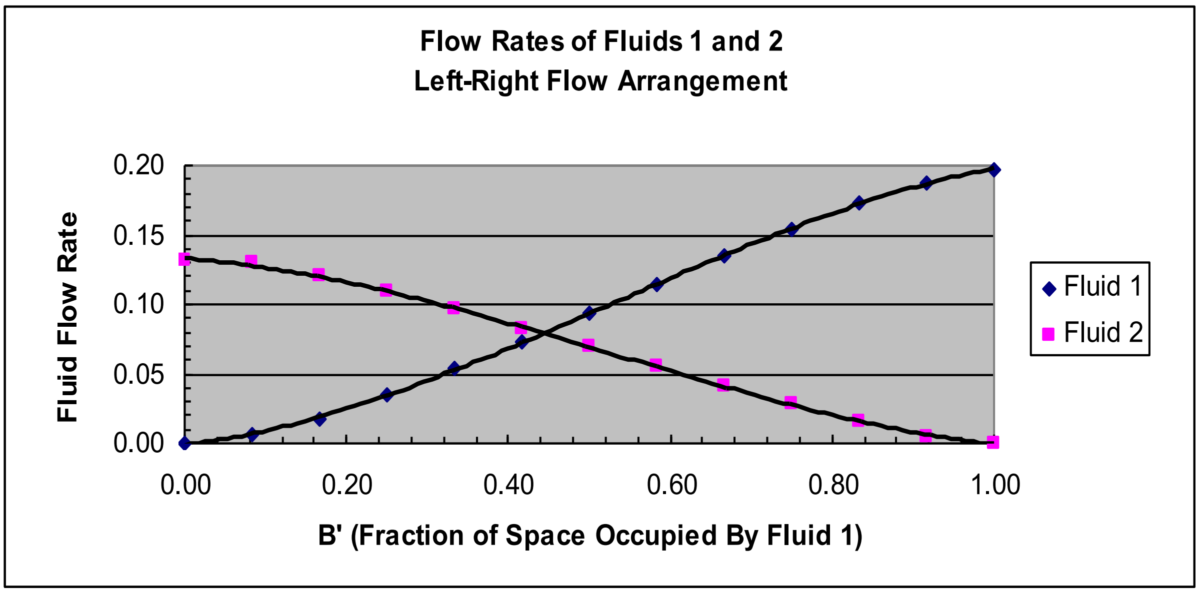


Figure 9. The flow rates of each fluid are plotted as a function of B' for the Left-Right configuration.

The value of B' where the flow rates are equal is seen to be approximately 0.44 from the graph. The value from linear interpolation is 0.4399.

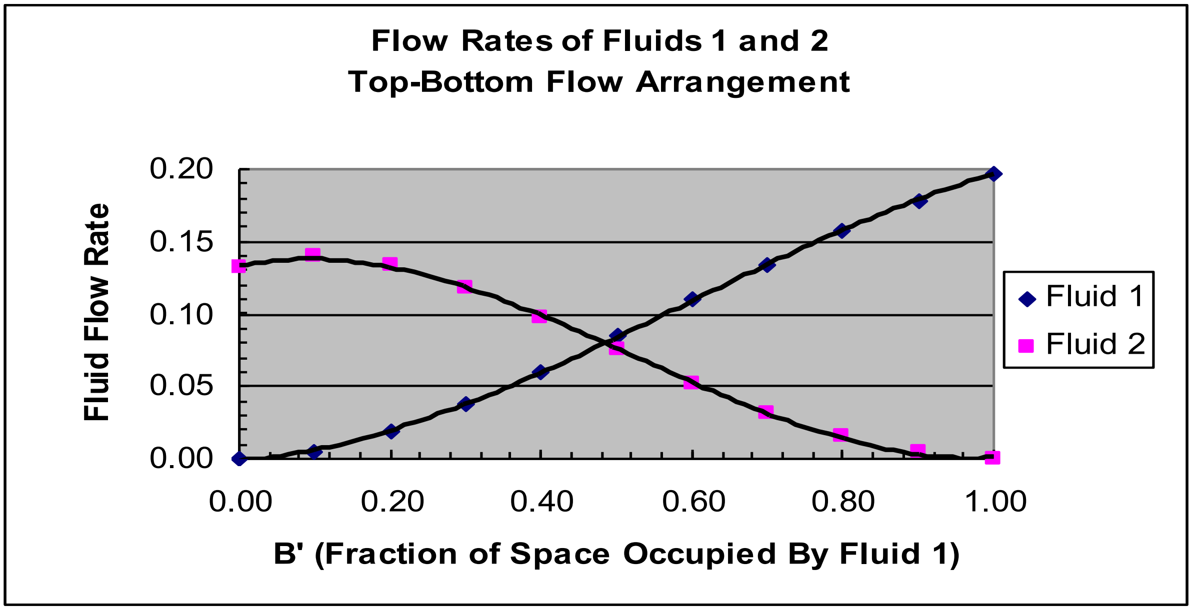


Figure 10. The flow rates of each fluid are plotted as a function of B' for the Top-Bottom configuration.

The value of B' where the flow rates are equal is seen to be approximately 0.48 from the graph. The value from linear interpolation is 0.4803.

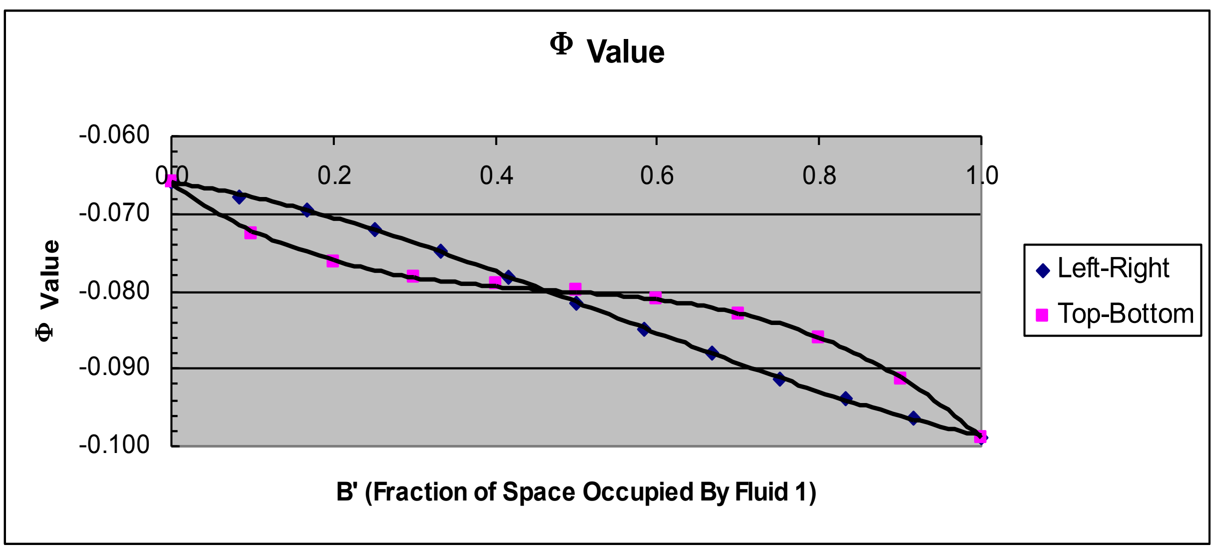


Figure 11. This plot shows the trend in the value of $\Phi$ for the Left-Right and Top-Bottom configurations.

Using the data presented in Table 3, the value of B* that makes the fluid flow rates equal for the Inside-Outside configuration can be estimated. The value of B* obtained using linear interpolation is 0.3016.

## Discussion

The difference in velocity profiles on the left and right sides of Figure 5 show the affect of adding more Fluid 1 to the channel. Most of the red coloring (high velocity) is concentrated on the left where the less-viscous Fluid 1 resides. The higher viscosity of Fluid 2 leads to an overall lower velocity and lower maximum velocity as shown by the coloring of the right side of the graph.

The difference in velocity profiles on the top and bottom sides of Figure 7 show the affect of adding more Fluid 1 to the channel in this configuration. Although the affect is not as great as for the Left-Right case, there is still a deviation from a pure fluid velocity profile and from Figure 6. The maximum velocity occurs just below the boundary in the Fluid 1 domain. Again, the lower viscosity of Fluid 1 means that it can achieve a higher velocity.

Figure 9 shows that the flow rate of each fluid increases steadily as the cross-sectional area of that fluid increases. Figure 10 shows a similar trend with one exception. For a B' value of zero, the channel contains entirely Fluid 2, which has the higher viscosity. Increasing B' is equivalent to adding a small slit of Fluid 1, which has a lower viscosity. The flow rate of Fluid 2 initially increases as the slit of Fluid 1 is added because Fluid 2 is now flowing over a material of lower viscosity that also happens to be flowing in the same direction. Eventually, the effect disappears as more Fluid 1 is added to the channel.

Figure 11 shows that the value of $\Phi$ decreases as the amount of Fluid 1 in the channel increases. Because the minimum value of $\Phi$ represents the preferred orientation, this trend makes sense. If the channel is filled entirely with the less viscous fluid, then a minimum in the amount of viscous dissipation is achieved.

The values of B' and B* that make the flow rates identical for each configuration can be used to find the value of the $\Phi$ using linear interpolation and the data from the tables. For the Left-Right configuration, the value of $\Phi$ is found to be –0.07904. For the Top-Bottom configuration, the value of $\Phi$ is –0.07977. For the Inside-Outside configuration, the value of $\Phi$ is –0.14448.

Linear interpolation is used for the Left-Right and Top-Bottom configurations in the regions of the graph where F appears linear in the neighborhood of equal flow rates. The linear assumption in this neighborhood is quite good.

Linear interpolation is used for the Inside-Outside configuration even though the value of $\Phi$ is clearly non-linear in B* (See Table 3) because an increase in B* increases the fraction of space occupied by Fluid 1 in a nonlinear fashion. The use of linear interpolation, though, does not affect the value of B* so much that the value of $\Phi$ could actually be higher than either of the $\Phi$ values obtained for the other configurations. The last entry of Table 3 supports this conclusion.

## Conclusion

The $\Phi$ value for the case of equal flow rates is lower for the Inside-Outside configuration. Therefore, it is concluded that the two fluids would take the Inside-Outside configuration if an identical experiment were performed in a laboratory.

## References

The following texts were studied during this research and deserve recognition for the enrichment that they provided to the author. Superscripts inserted into the text indicate a literature citation.


1. Bird, R. Byron, Edwin N. Lightfoot, and Warren E. Stewart. Transport Phenomena. John Wiley and Sons: New York, 1960.

2. Bliss, Gilbert Ames. Calculus of Variations. The Open Court Publishing Company: La Salle, 1971.

3. Finlayson, Bruce A. The Method of Weighted Residuals and Variational Principles. Academic Press: New York, 1972.