THE PHYSICS OF FLUIDS VOLUME 15, NUMBER 6 JUNE 1972

Existence of Variational Principles for the Navier–Stokes Equation

BRUCE A. FINLAYSON
Department of Chemical Engineering, University of Washington, Seattle, Washington 98195
(Received 2 September 1971; final manuscript received 20 December 1971)

Fréchet differentials are introduced to decide when a classical variational principle exists for a given nonlinear differential equation. The formalism is applied to the steady-state Navier–Stokes equation and the continuity equation, and no variational principle exists unless $\mathbf{u}\times(\nabla\times\mathbf{u})=0$ or $\mathbf{u}\cdot\nabla\mathbf{u}=0$. The concept of an adjoint equation is extended to nonlinear equations and a variational principle is derived for the Navier–Stokes equation and its adjoint.

I. INTRODUCTION

Variational principles may succinctly summarize equations of motion, allow insight into the effect of parameters, and provide a means for approximating the solution. For some problems minimum and maximum principles may hold, leading to reciprocal variational principles which provide a means for obtaining upper and lower bounds on a variational integral. In other problems, however, the variational principle is only a stationary principle and no minimum or maximum is achieved. In still other problems a variational principle may not hold at all. Given a differential equation and boundary conditions, it is an important question to decide if a variational principle exists for that problem. The search for variational principles encompassing fluid mechanics provides a varied history. The first principle, by Lord Kelvin,[1] applies to an incompressible, inviscid fluid in irrotational flow. The variational principle minimizes the kinetic energy, and a reciprocal principle holds as well.[2,3] If the fluid is compressible, but still in steady-state motion, the Bateman–Dirichlet principle holds,[4] and reciprocal variational principles are applicable, with the variational integral being the kinetic energy plus one-half the pressure. Various modifications of this principle occur for various equations of state and in infinite regions.[2,3]

The time-dependent equations of an inviscid, compressible fluid also have variational principles. The Lagrangian form of the equations is treated by Herivel[5] and Eckart,[6] and the variational integral is the kinetic energy minus the sum of internal and potential energy. If the equations are treated in their Eulerian form, the variational principle is due to Herivel,[5] when modified using Lin's constraint to permit rotational flows.[2,7] Eckart[8] had earlier presented such a principle for irrotational flows when the internal energy depends only on the density, and Seliger and Whitham[9] derive a principle which incorporates some of the constraints in the representation for velocity. Stephens[10] considers the equations in canonical form for the energy-momentum tensor, while Penfield[11] treats the equations in relativistic form.

In all of the above principles, the fluid is inviscid. The inclusion of Newtonian viscous terms in the equations of motion leads to difficulties except in special cases. If the inertial terms are absent, then the Helmholtz–Korteweg principle applies to the steady-state flow of an incompressible fluid, and the viscous dissipation is minimized in the variational principle.[12] If the fluid is non-Newtonian, variational principles may hold,[13,3] but still when the inertial terms are absent in the equations.

Thus, we have the interesting situation that variational principles exist when inertial terms are important and viscous terms are not and when viscous terms are important but inertial terms are not. Attempts to derive variational principles when both inertial and viscous terms are included have so far failed. Brill[14] noticed that the Helmholtz–Korteweg principle could be generalized to include in the equations the inertial term $\nabla(\frac{1}{2}\mathbf{u}\cdot\mathbf{u})$, but not the full inertial term,

$$\mathbf{u}\cdot\nabla\mathbf{u}=\nabla(\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u})-\mathbf{u}\times(\nabla\times\mathbf{u}). \tag{1}$$

Millikan[15] gave the definitive treatment of the existence of a variational principle for the steady-state Navier–Stokes equation for an incompressible fluid. By means of a detailed, lengthy argument, he concluded that a variational principle could not be found unless $\mathbf{u}\cdot\nabla\mathbf{u}=0$ or $\mathbf{u}\times(\nabla\times\mathbf{u})=0$. By the use of Fréchet differentials, we are able to shorten Millikan's proof considerably, as well as illustrate a general method for deciding if a differential equation can be derived from a variational principle. We then extend the concept of an adjoint equation to nonlinear equations, and derive a variational principle for the steady-state Navier–Stokes equation and its adjoint equation.

II. FRÉCHET DERIVATIVES

Fréchet derivatives are introduced to help answer the question of whether a differential equation is the Euler–Lagrange equation of some variational principle. The discussion follows and extends that of Tonti[16,17] although the important theorem was proved by Vainberg.[18]

Consider the analogy of a vector field. A vector $\mathbf{v}$ is derivable from a potential $\mathbf{v}=\nabla\phi$ if $\nabla\times\mathbf{v}=0$. If this is true, then the integral

$$\int_1^2 \mathbf{v}\cdot d\mathbf{x}=\phi_2-\phi_1 \tag{2}$$

depends only on the end points, not on the path of integration. Equation (2) then provides a method of testing a vector field to see if it is the gradient of a scalar function. The integration is performed over two different, infinitesimal paths between two points, and if the integral is independent of the path, the velocity is the gradient of a scalar function.

Next consider a nonlinear differential operator

$$N(u)=0.$$

The Fréchet differential in the direction ϕ is given by

$$N_u'\phi=\lim_{\epsilon\to 0}\frac{N(u+\epsilon\phi)-N(u)}{\epsilon}=\left[\frac{\partial}{\partial\epsilon}N(u+\epsilon\phi)\right]_{\epsilon=0}, \quad (3)$$

where N_u' is called the Fréchet derivative (taken with respect to u). The gradient of a functional is defined similarly. Given a functional

$$F(u)=\int L(u)\,dV$$

the Fréchet differential in the direction ϕ is given by

$$\lim_{\epsilon\to 0}\{[F(u+\epsilon\phi)-F(u)]/\epsilon\}=\int L_u'\phi\,dV.$$

Integration by parts to remove the derivatives operating on ϕ gives

$$\lim_{\epsilon\to 0}\{[F(u+\epsilon\phi)-F(u)]/\epsilon\}=\int L_u'\phi\,dV$$
$$=\int\phi N(u)\,dV+\text{boundary terms.} \quad (4)$$

The operator $N(u)$ is the gradient of the functional $F(u)$.

To test if an operator $N(u)$ is the gradient of a functional we must see if the integration in Eq. (4) is the same when taken over the two infinitesimal paths.

(a) $u\to u+\epsilon\phi\to u+\epsilon\phi+\nu\psi$

(b) $u\to u+\nu\psi\to u+\epsilon\phi+\nu\psi.$

Thus, we need

$$\int N(u)\epsilon\phi\,dV+\int N(u+\epsilon\phi)\nu\psi\,dV$$
$$=\int N(u)\nu\psi\,dV+\int N(u+\nu\psi)\epsilon\phi\,dV$$

Dividing by $\epsilon\nu$ and taking the limit as ϵ, $\nu\to 0$ give

$$\int\psi N_u'\phi\,dV=\int\phi N_u'\psi\,dV. \quad (5)$$

If Eq. (5) holds, the operator N_u' is symmetric, and Vainberg[18] proves that this is a sufficient condition for the operator $N(u)$ to be the gradient of a functional. The functional itself is (except for possible boundary terms)

$$F(u)=\int u\int_0^1 N(\lambda u)\,d\lambda\,dV \quad (6)$$

and the variation can be shown to be[16,3]

$$\delta F=\int N(u)\delta u\,dV.$$

The fact that $N(u)$ is the Euler–Lagrange equation for a variational principle with $F(u)$ as the functional is readily apparent from the last equation.

Next consider the vector differential equation

$$f^l(u_s;u_{s,j};u_{s,jk})=0$$

with $u_s=u_s(x_1,x_2,x_3)$. The subscripts or superscripts represent Cartesian components and a comma denotes differentiation. Thus,

$$u_{s,j}=\frac{\partial u_s}{\partial x_j}.$$

It is shown elsewhere[16,3] that the conditions for the analogous equation (5) to hold, and hence for the operator f^l to be the gradient of a functional, are

$$\frac{\partial f^l}{\partial u_{s,jk}}=\frac{\partial f^s}{\partial u_{l,jk}}, \quad (7)$$

$$\frac{\partial f^l}{\partial u_{s,j}}=-\frac{\partial f^s}{\partial u_{l,j}}+2\nabla_k\frac{\partial f^s}{\partial u_{l,jk}}, \quad (8)$$

$$\frac{\partial f^l}{\partial u_s}=\frac{\partial f^s}{\partial u_l}-\nabla_j\frac{\partial f^s}{\partial u_{l,j}}+\nabla_j\nabla_k\frac{\partial f^s}{\partial u_{l,jk}}. \quad (9)$$

To test a differential equation, then, we merely evaluate Eqs. (7)–(9) to see if they hold. If they do, then a variational principle exists for the differential equation. We next apply this procedure to the Navier–Stokes equations.

III. NAVIER–STOKES EQUATION

We consider the steady-state Navier–Stokes equation for an incompressible fluid. Thus, we assume the fluid is Newtonian:

$$\rho\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla p+\mu\nabla^2\mathbf{u}, \quad (10)$$

$$\nabla\cdot\mathbf{u}=0. \quad (11)$$

Forces derivable from a potential can be included in the pressure term. We know that if either $\mu=0$ or $\mathbf{u}\cdot\nabla\mathbf{u}=0$ a variational principle is possible. We use Fréchet differentials to decide if a variational principle can be found when neither of these conditions holds, although we first apply the formalism in the case when $\mathbf{u}\cdot\nabla\mathbf{u}=0$.

Consider the four-vector $\mathbf{w}=(\mathbf{u},p)$ and the four equations

$$f^\alpha=-w_{4,\alpha}+\mu w_{\alpha,\beta\beta},$$

$$f^4=w_{\beta,\beta}=0. \quad (12)$$

Greek indices run from one to three and Latin indices go from one to four; repeated indices are summed over their range. If a Fréchet differential is to exist for the set (12), it is necessary that Eqs. (7)–(9) be satisfied.

The derivatives are

$$\frac{\partial f^\alpha}{\partial w_{\beta,\gamma\delta}}=\mu\delta_{\alpha\beta}\delta_{\gamma\delta}, \qquad \text{zero otherwise,}$$

$$\frac{\partial f^4}{\partial w_{\alpha,\beta}}=\delta_{\alpha\beta}, \qquad \frac{\partial f^\alpha}{\partial w_{4,\beta}}=-\delta_{\alpha\beta}, \qquad \text{zero otherwise,}$$

$$\frac{\partial f^l}{\partial w_s}=0.$$

Equation (7) is easily satisfied. Equation (8) is

$$\frac{\partial f^4}{\partial w_{\alpha,\beta}}=\delta_{\alpha\beta}=-\frac{\partial f^\alpha}{\partial w_{4,\beta}}=-(-\delta_{\alpha\beta})$$

and is zero for other indices; it is thus satisfied. Equation (9) is satisfied since all terms are zero. The functional is determined from Eq. (6) to be $\frac{1}{2}w_i f^i$ which gives

$$J=-\tfrac{1}{2}\int\mu u_{\alpha,\beta}u_{\alpha,\beta}\,dV+\int p u_{\gamma,\gamma}\,dV+\text{boundary terms.}$$

This is the variational integral in the Helmholtz–Korteweg principle with a Lagrange multiplier introduced to account for the constraint $\nabla\cdot\mathbf{u}=0$. The functional J is minimized; thus the viscous dissipation is minimized.

The same result applies if w_4 is interpreted as $w_4=p+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}$, giving Brill's result.[14]

Next consider the full equations (10).

$$f^\alpha=-w_{4,\alpha}+\mu w_{\alpha,\beta\beta}+A\rho\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\delta\epsilon}w_\beta w_{\epsilon,\delta},$$
$$f^4=w_{\beta,\beta}=0. \tag{13}$$

We take $w_4=p+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}$ and A is a constant, introduced for convenience, which takes the values 0 or 1. The various derivatives are

$$\frac{\partial f^\alpha}{\partial w_{\beta,\gamma\delta}}=\mu\delta_{\alpha\beta}\delta_{\gamma\delta}, \qquad \text{zero otherwise,}$$

$$\frac{\partial f^\alpha}{\partial w_{\beta,\gamma}}=A\rho\epsilon_{\alpha\delta\epsilon}\epsilon_{\epsilon\gamma\beta}w_\delta,$$

$$\frac{\partial f^\alpha}{\partial w_{4,\gamma}}=-\delta_{\alpha\gamma}, \qquad \frac{\partial f^4}{\partial w_{\beta,\gamma}}=\delta_{\beta\gamma}, \frac{\partial f^4}{\partial w_{4,\gamma}}=0,$$

$$\frac{\partial f^\alpha}{\partial w_\beta}=A\rho\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\delta\epsilon}w_{\epsilon,\delta}, \qquad \text{zero otherwise.}$$

Equation (8) gives, for l, $s=1$, 2, or 3

$$A\rho(\epsilon_{\alpha\delta\epsilon}\epsilon_{\epsilon\gamma\beta}+\epsilon_{\beta\delta\epsilon}\epsilon_{\epsilon\gamma\alpha})w_\delta=0.$$

If we take $\alpha=1$ and $\beta=2$, then ϵ must be 3, giving

$$A\rho(\epsilon_{1\delta3}\epsilon_{3\gamma2}w_\delta+\epsilon_{2\delta3}\epsilon_{3\gamma1}w_\delta)=0,$$

or $Aw_1=Aw_2=0$. $\beta=3$ gives $Aw_3=0$. Thus, we obtain

$$A\mathbf{u}=0$$

as the equation which must be satisfied if the Fréchet differential is symmetric and a variational principle exists. This gives only a trivial result unless $A=0$. We conclude then that a variational principle exists for the steady-state Navier–Stokes equations (10) only if $\mathbf{u}\times(\nabla\times\mathbf{u})=0$.

If the inertial term is written in the form $\mathbf{u}\cdot\nabla\mathbf{u}$ and we apply the conditions Eqs. (7)–(9), we find that no variational principle exists unless $\mathbf{u}\cdot\nabla\mathbf{u}=0$.

We next consider the introduction of a weighting factor, since this sometimes changes an operator to a symmetric form.[16]

$$f^\alpha=g(w_m,w_{m,n})[-w_{4,\alpha}+\mu w_{\alpha,\beta\beta}+A\rho\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\delta\epsilon}w_\beta w_{\epsilon,\delta}], \tag{14}$$

$$f^4=h(w_m,w_{m,n})w_{\alpha,\alpha}. \tag{15}$$

Now the various derivatives are

$$\frac{\partial f^\alpha}{\partial w_{\beta,\gamma\delta}}=\mu g\delta_{\alpha\beta}\delta_{\gamma\delta}, \qquad \text{zero otherwise,}$$

$$\frac{\partial f^\alpha}{\partial w_{\beta,\gamma}}=\frac{f^\alpha}{g}\frac{\partial g}{\partial w_{\beta,\gamma}}+A\rho g\epsilon_{\alpha\eta\nu}\epsilon_{\nu\gamma\beta}w_\eta,$$

$$\frac{\partial f^\alpha}{\partial w_{4,\gamma}}=-g\delta_{\alpha\gamma}+\frac{f^\alpha}{g}\frac{\partial g}{\partial w_{4,\gamma}},$$

$$\frac{\partial f^4}{\partial w_{\beta,\gamma}}=h\delta_{\beta\gamma}+w_{\alpha,\alpha}\frac{\partial h}{\partial w_{\beta,\gamma}},$$

$$\frac{\partial f^4}{\partial w_{4,\gamma}}=w_{\alpha,\alpha}\frac{\partial h}{\partial w_{4,\gamma}},$$

$$\frac{\partial f^\alpha}{\partial w_\beta}=A\rho g\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\delta\epsilon}w_{\epsilon,\delta}+\frac{f^\alpha}{g}\frac{\partial g}{\partial w_\beta},$$

$$\frac{\partial f^\alpha}{\partial w_4}=\frac{f^\alpha}{g}\frac{\partial g}{\partial w_4}, \qquad \frac{\partial f^4}{\partial w_i}=w_{\alpha,\alpha}\frac{\partial h}{\partial w_i}.$$

Now Eq. (8) requires, for $l=s=1$,

$$A\rho g\epsilon_{1\eta\nu}\epsilon_{\nu\gamma1}w_\eta+\frac{f^1}{g}\frac{\partial g}{\partial w_{1,\gamma}}=\mu\nabla_\delta(g\delta_{\gamma\delta})$$
$$=\mu\left(\frac{\partial g}{\partial w_m}w_{m,\gamma}+\frac{\partial g}{\partial w_{m,\epsilon}}w_{m,\epsilon\gamma}\right). \tag{16}$$

Since g is independent of $w_{m,\epsilon\gamma}$, and Eq. (16) must hold for all functions w_m it is necessary that $\partial g/\partial w_{m,\epsilon}=0$. Then, g depends only on w_m. However, Eq. (16) requires the coefficient of $w_{m,\gamma}$ to be zero; i.e., $\partial g/\partial w_m=0$. Thus, g is a constant. Similarly, Eq. (8) for $l=4$, $s=\beta$, $j=\gamma$ gives

$$h\delta_{\beta\gamma}+w_{\alpha,\alpha}\frac{\partial h}{\partial w_{\beta,\gamma}}=-(-g\delta_{\beta\gamma}).$$

Choosing $\beta \neq \gamma$ gives $\partial h/\partial w_{\beta,\gamma}=0$ and then $h=g$. Thus, both g and h are constants and no improvement results from the weighting function.

We have shown that Eq. (13) is the Euler–Lagrange equation of a variational principle only for $A=0$, i.e., $\mathbf{u}\times\nabla\times\mathbf{u}=0$. This result holds for equations in the form of Eq. (13) even if $\mu=0$. For an inviscid fluid, however, we know the inertial terms can be included in a variational principle. The resolution of this paradox lies in the form of the equations. For the inviscid fluid consider the equations and variables ordered as follows.

$$
\begin{aligned}
(w_1, w_2, w_3) &= \mathbf{u}, & \rho\mathbf{u}+\rho\nabla\alpha-\rho\beta\nabla S-\rho\gamma\nabla a&=0,\\
w_4 &= \rho, & \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}-U-\Phi-p/\rho+\mathbf{u}\cdot\nabla\alpha&\\
& & -\beta\mathbf{u}\cdot\nabla S-\gamma\mathbf{u}\cdot\nabla a&=0,\\
w_5 &= S, & -\rho T+\rho\mathbf{u}\cdot\nabla\beta+\beta\rho\nabla\cdot\mathbf{u}+\beta\mathbf{u}\cdot\nabla p&=0,\\
w_6 &= a, & \rho\mathbf{u}\cdot\nabla\gamma+\gamma\rho\nabla\cdot\mathbf{u}+\gamma\mathbf{u}\cdot\nabla\rho&=0, \quad (17)\\
w_7 &= \alpha, & -\rho\nabla\cdot\mathbf{u}-\mathbf{u}\cdot\nabla\rho&=0,\\
w_8 &= \beta, & -\rho\mathbf{u}\cdot\nabla S&=0,\\
w_9 &= \gamma, & -\rho\mathbf{u}\cdot\nabla a&=0.
\end{aligned}
$$

Then, the application of Eqs. (7)–(9) shows that a variational principle should exist for this system of equations. The variational integral is, of course, the usual one.[4]

The use of Fréchet differentials gives a straightforward method of deciding if a set of equations has a variational principle. The formalism, and results, depend on the form of the equations. A variational principle holds for Eq. (17), but not for Eq. (13) with $\mu=0$, $A=1$, even though both sets of equations refer to the same physical problem.

IV. ADJOINT VARIATIONAL PRINCIPLE

For linear self-adjoint problems a variational principle exists. For linear, nonself-adjoint problems a variational principle exists for the problem and its adjoint.[19] We define an adjoint equation for nonlinear operators, and use this concept to derive a variational principle for the Navier–Stokes equation and its adjoint.

Replace Eq. (5) with

$$\int \psi N_u'\phi\, dV=\int \phi\tilde{N}_u'\psi\, dV+\int B(\chi,\phi)\, dS, \quad (18)$$

where the left-hand side is integrated by parts to eliminate derivatives operating on ϕ and the term involving B comes from the integration by parts. The term B is evaluated on the boundary and the exact form depends on the differential operator $N(u)$.

The right-hand side then defines the Fréchet derivative, $\tilde{N}_u'$, which is adjoint to N_u'. We then define the adjoint operator

$$N^*(u,v)\equiv\tilde{N}_u'v. \quad (19)$$

If $N_u'=\tilde{N}_u'$ the Fréchet differential is symmetric and a variational principle exists. If not, we can construct a variational principle for $N(u)$ and $N^*(u,v)$ together from the functional

$$I(u,v)=\int vN(u)\, dV.$$

The Euler–Lagrange equations are then $N(u)=0$ and $N^*(u,v)=0$, which can be seen by taking the variation with respect to both u and v and using Eq. (18). We next apply this procedure to the Navier–Stokes equations.

The variational principle is generated by considering the functional

$$I(w_i, w_i^*)=\int w_i^* f^i\, dV, \quad (20)$$

where f^α is given by Eq. (10) and f^4 by Eq. (11). Integration of Eq. (20) by parts gives the adjoint equation, and addition of appropriate boundary terms accounts for natural boundary conditions. We can thus be led to the following principle:

Make $I(u,u^*)$ stationary subject to the constraints

$$w_{\alpha,\alpha}=0, \qquad w_{\alpha,\alpha}^*=0 \text{ in } V, \quad (21)$$

$$w_\alpha=g_\alpha, \qquad w_\alpha^*=g_\alpha^* \text{ on } S_1, \quad (22)$$

$$w_\alpha n_\alpha=g, \qquad w_\alpha^* n_\alpha=g^* \text{ on } S_2, \quad (23)$$

$$
\begin{aligned}
I(u,u^*)=&\int_v [\rho w_\alpha^* w_\beta w_{\alpha,\beta}+2\mu\, d_{\gamma\beta}\, d_{\gamma\beta}^*]\, dV\\
&-\int_{s3} g_\beta w_\beta^*\, dS-\int_{s2} g_\beta(w_\beta^*-w_\gamma^* n_\gamma n_\beta)\, dS\\
&-\int_{s3} g_\beta^* w_\beta\, dS-\int_{s2} g_\beta^*(w_\beta-w_\gamma n_\gamma n_\beta)\, dS.
\end{aligned}
$$

In these equations we use

$$d_{\alpha\beta}=\tfrac{1}{2}(w_{\alpha,\beta}+w_{\beta,\gamma}), \qquad T_{\alpha\beta}=2\mu d_{\alpha,\beta},$$

for both primary (w_α) and adjoint (w_α^*) variables. The functions, g, g^*, g_β, g_β^* are specified functions of position on the boundary.

The incompressibility constraints are included by introducing Lagrange multipliers λ^* and λ, and subtracting from the variational integral the term

$$\int_v [\lambda^* w_{\alpha,\alpha}+\lambda w_{\alpha,\alpha}^*]\, dV.$$

Variations with respect to w_α, w_α^*, λ^*, and λ then give the Euler equations

$$\delta w_\alpha:\quad \rho w_\beta w_{\alpha,\beta}+\lambda_{,\alpha}-2\mu d_{\alpha\beta,\beta}=0, \quad (24)$$

$$\delta w_\alpha^*:\quad \rho w_\beta^* w_{\beta,\alpha}-\rho(w_\alpha^* w_\beta)_{,\beta}-2\mu d_{\alpha\beta,\beta}^*+\lambda_{,\alpha}^*=0, \quad (25)$$

$$\delta\lambda^*:\quad w_{\alpha,\alpha}=0, \qquad \delta\lambda:\quad w_{\alpha,\alpha}^*=0.$$

Equation (24) is, of course, the Navier–Stokes equation, while Eq. (25) is the adjoint to it. Note that the adjoint operator depends on both primary and adjoint variables, but is linear in the adjoint variables. This is a feature of the way it was derived using Eq. (18). The Lagrange multipliers λ and λ^* can be interpreted as the pressures. The essential boundary conditions are Eq. (22)–(23) while the natural boundary conditions are

$$\left.\begin{array}{l} 2\mu n_\gamma d_{\beta,\gamma}-\lambda n_\beta - g_\beta = 0 \\ \text{or} \\ n_\gamma T_{\gamma\beta}-\lambda n_\beta = g_\beta \end{array}\right\} \text{ on } S_3,$$

$$\left.\begin{array}{l} 2\mu(d_{\alpha\beta}n_\beta - d_{\gamma\beta}n_\beta n_\gamma n_\alpha) - g_\beta = 0 \\ \text{or} \\ n_\beta T_{\beta\alpha} - T_{\beta\gamma}n_\beta n_\gamma n_\alpha = g_\beta \end{array}\right\} \text{ on } S_2,$$

$$\left.\begin{array}{l} 2\mu n_\beta d_{\alpha\beta}^* + \rho w_\alpha^* w_\beta n_\beta - \lambda^* n_\alpha - g_\alpha^* = 0 \\ \text{or} \\ n_\beta T_{\alpha\beta}^* + \rho w_\alpha^* w_\beta n_\beta - \lambda^* n_\alpha = g_\alpha^* \end{array}\right\} \text{ on } S_3,$$

$$\left.\begin{array}{l} 2\mu(d_{\alpha\beta}^* n_\beta - d_{\gamma\beta}n_\beta n_\gamma n_\alpha) \\ \quad + \rho w_\alpha n_\alpha (w_\beta^* - w_\gamma^* n_\gamma n_\beta) - g_\beta^* = 0 \\ \text{or} \\ n_\beta T_{\beta\alpha}^* - T_{\beta\gamma}^* n_\beta n_\gamma n_\alpha \\ \quad + \rho w_\alpha n_\alpha (w_\beta^* - w_\gamma^* n_\gamma n_\beta) = g_\beta^* \end{array}\right\} \text{ on } S_2.$$

The boundary conditions on S_2 are determined by writing, e.g.,

$$\delta w_\alpha^* = (\delta w_\alpha^* - \delta w_\beta^* n_\beta n_\alpha) + \delta w_\beta^* n_\beta n_\alpha$$

and then noting that $\delta(w_\beta^* n_\beta)=0$ on S_2 by condition (23) and only the tangential component, $\delta w_\alpha^* - \delta w_\beta^* n_\beta n_\alpha$, is an arbitrary variation on S_2.

We note the problem involving w_α is just the Navier–Stokes equation with general boundary conditions to permit the velocity to be specified on S_1, the normal component of velocity and tangential stress are specified on S_2, and the stress is specified on S_3. The functions g^* and g_β^* are specified functions chosen at will, since the adjoint problem has no physical meaning. In other linear problems these nonhomogeneous functions can be chosen in such a way that the variational integral represents some quantity of physical interest.[19] Here, that does not appear to be possible. We have thus constructed a variational principle for the Navier–Stokes equation and its adjoint, Eq. (25). The result is formal, however, because the adjoint equation has no physical interpretation. Slattery[20] used an approach related to that used here, in that he introduced nonlinear adjoint equations, but the use of Fréchet differentials aids in the choice of variational integral and makes systematic the question of the existence and the generation of the variational principle.

V. CONCLUSIONS

Fréchet differentials can be used to answer the question of the existence of a variational principle for nonlinear operators. The Navier–Stokes equation cannot be derived from a classical variational principle involving only the velocity unless one of the terms $\mathbf{u}\times(\nabla\times\mathbf{u})$ or $\mathbf{u}\cdot\nabla\mathbf{u}$ is zero. An adjoint equation can be defined for nonlinear operators using Fréchet differentials. A variational principle can then be found for the Navier–Stokes equation and its adjoint equation.

ACKNOWLEDGMENT

This research was supported by the National Science Foundation Grant No. GK-12517.

[1] W. Thomson, Cambridge Dublin Math. J. **4**, 90 (1849).

[2] See J. Serrin, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 8, Pt. 1, p. 125, for a detailed statement of the variational principles listed here.

[3] B. A. Finlayson, *The Method of Weighted Residuals and Variational Principles* (Academic, New York, 1972), Chaps. 8 and 9, gives a summary of variational principles applied to fluid mechanics and additional details on the mechanics of applying Fréchet differentials.

[4] See H. Bateman, Proc. Roy. Soc. (London) **A125**, 598 (1929); Proc. Natl. Acad. Sci. U.S. **16**, 816 (1930); and P. E. Lush and T. M. Cherry, Quart. J. Mech. Appl. Math. **9**, 6 (1956).

[5] J. W. Herivel, Proc. Cambridge Phil. Soc. **51**, 344 (1955).

[6] C. Eckart, Phys. Fluids **3**, 421 (1960).

[7] See F. P. Bretherton, J. Fluid Mech. **44**, 19 (1970) for an interpretation of the geometric coordinates introduced by Lin's constraint.

[8] C. Eckart, Phys. Rev. **54**, 920 (1938).

[9] R. L. Seliger and G. B. Whitham, Proc. Roy. Soc. (London) **A305**, 1 (1968).

[10] J. J. Stephens, Phys. Fluids **10**, 76 (1967).

[11] P. Penfield, Jr., Phys. Fluids **9**, 1184 (1966).

[12] The theorem was first stated by Helmholtz in 1882 [see H. Lamb, *Hydrodynamics* (Dover, New York, 1945), 6th., p. 617] and proved to be an absolute minimum by D. G. Korteweg, Phil. Mag. **16**, 112 (1883). Reciprocal principles were provided by R. Hill and G. Power, Quart. J. Mech. Appl. Math. **9**, 313 (1956), and H. Lippman, Ing. Arch. **32**, 347 (1963).

[13] See, e.g., M. W. Johnson, Jr., Phys. Fluids **3**, 871 (1960); Trans. Soc. Rheol. **5**, 9 (1961).

[14] J. Brill, Proc. Cambridge Phil. Soc. **8**, 313 (1895). This was noticed by Lord Rayleigh, too, in Phil. Mag. **26**, 776 (1913).

[15] C. B. Milliken, Phil. Mag. **7**, 641 (1929).

[16] E. Tonti, Bull. Classe Sci. Acad. Roy. Belg. (5) **55**, 137, 262 (1969).

[17] See also Ref. 3.

[18] M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators* (Holden-Day, San Francisco, Calif., 1964), Chap. 2.

[19] J. Lewins, *Importance: The Adjoint Function* (Pergamon, New York 1965), Chap. 1. See also, e.g., P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. I, p. 313, which gave a variational principle for the unsteady heat conduction equation, and P. Roussopoulos, Compt. Rend. **236**, 1858 (1953), who gave a variational principle for any linear differential equation and its adjoint.

[20] J. C. Slattery, Chem. Eng. Sci. **19**, 801 (1964).